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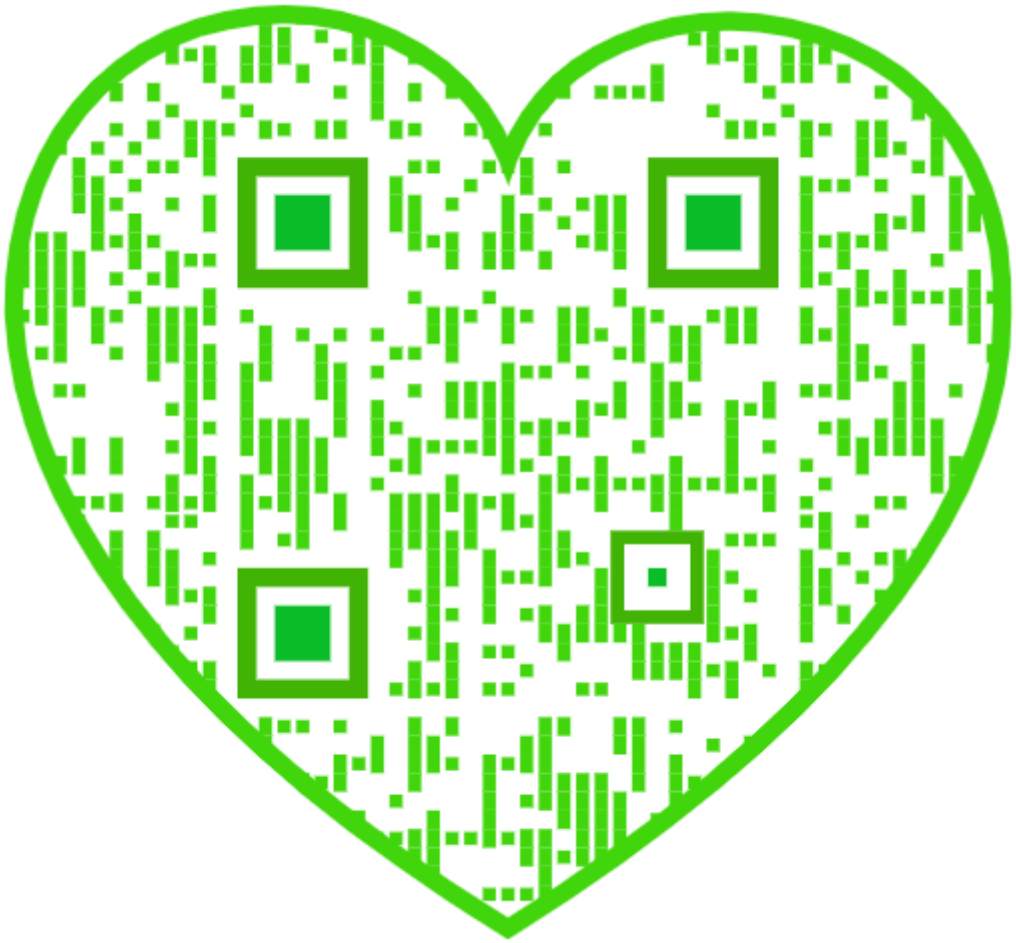

DESCRIPTIVE GEOMETRY

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DESCRIPTIVE GEOMETRY

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PREFACE

The book "Notes on Descriptive Geometry" by William L. Ames, was first published in 1893, and was probably the earliest book on the subject which used the third quadrant. Even today, when the third quadrant is used exclusively in the drafting offices of this country, the subject of Descriptive Geometry is to a considerable extent taught as first quadrant projection.

The present book is in reality little more than an enlargement of Professor Ames' book, the principal changes being in the increased number of exercises. Additional illustrations have been put in and a few chapters added. In this work of revision and enlargement the author desires to express his indebtedness to Professor John B. Peddle of the Rose Polytechnic Institute.

C. W.

Terre Haute, Ind.
Sept. 1, 1918.

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DESCRIPTIVE GEOMETRY

I. Introduction.

Descriptive Geometry treats of methods of representing on plane surfaces, magnitudes of three dimensions in such a way that their forms and positions may be completely determined; and conversely, of methods of determining the forms and positions of magnitudes thus represented.

If the intersection with a plane of the rays from all points of an object to the observer is taken as the representation of the object on the plane, the production of such a representation requires that three things be known, namely the position of the object, of the plane of representation and of the observer.

In Fig. 1, let p represent a point in space and o the position of the observer. A line drawn from one to the other will intersect the vertical plane at p^v, and we say that p^v is the projection of the point p upon this plane. In like manner suppose the position of the observer to be shifted to o' and again connected to p by a straight line. The intersection of the horizontal plane and this line gives us the point p^h, which is called the projection of p upon the horizontal plane.

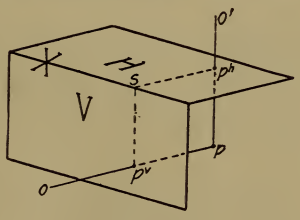


FIG. 1.

Thus knowing the position of the point p in space with reference to the planes of representation, and the position of the observer, or the direction of a line from the point to

the observer, we are able to find the projections of the point.

Or, if we reverse the process and have given the point p^v as the projection of the point p upon the vertical plane, when the observer is at o, we know that p must lie somewhere upon the line op^v extended, though its exact position on this line is indeterminate. Also if p^h is the projection of p upon the horizontal plane, when o' is the observer's position, we know that p lies somewhere upon $o'p^h$ extended. Now having p^v and p^h with the directions op^v and $o'p^h$, the location of the point p in space is fully determined with reference to the planes of representation.

This method is that used in Descriptive Geometry, but simplified by the following modifications: The planes of representation, which are also used as planes of reference as to position, are taken at right angles to each other and are considered as unlimited in extent. The points of observation are taken to be at an infinite distance away, so that the rays become lines perpendicular to the planes of projection and the representations are called orthographic projections of the object in space.

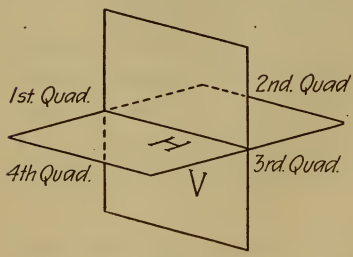


FIG. 2.

One of the planes is taken vertical and called the vertical plane of projection or V; the other is taken horizontal and called the horizontal plane of projection or H. The two planes intersect in a line called the ground line and designated by X. The intersection of the two planes forms four dihedral angles called quadrants. The space above H and in front of V is called the first quadrant, above H and behind V the second, below H and behind V the third, and below H and in front of V the fourth (Fig. 2).

For the convenience of the draftsman the planes are considered as rotated about X, so that the first and third quadrants open to 180° and the second and fourth quadrants close to 0°.

II. Representation of Points.

In considering the representation of a point as modified by the above conditions, X is taken as a horizontal line, and indicates the line of intersection of H and V. It also represents the axis about which H and V are rotated into one plane which is represented by the plane of the paper. All the representations or projections which, when the planes are at right angles, would be in V above H or in H back of V, will appear in the drawing above X. All projections which would be in V below H or in H in front of V, will appear in the drawing below X. Hence it will be seen that if a point is in the first quadrant its projection on the V plane, called its V projection or elevation, will be above X, and its projection on the H plane, called its H projection or plan, will be below X. If the point lies in the second quadrant, both projections will be above X; if in the third quadrant, the H projection will be above X and the V projection below; if in the fourth quadrant, both projections will be below X.

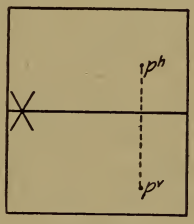


FIG. 3.

From Fig. 1 it is seen that since $p^v p$ and $p^h p$ are respectively perpendicular to V and H, that the projection of all points of the line $p^h p$ on V will be the line sp^v perpendicular to the line of intersection of H and V. Also the projection of the line $p^v p$ on H will be the line sp^h also perpendicular to the line of intersection of H and V. Hence when H and V are revolved into the same plane, p^v and p^h will lie in the same perpendicular to X, as in Fig. 3. The distance from p^h to X measures the distance from p to the V plane, and the

distance from p^v to X measures the distance from p to the H plane.

Figure 4 shows the projections or representations of four points, one lying in each quadrant. From this it will be noticed that any two points lying in the same perpendicular to X may be taken as the projections of a point in space.

NOTATION.—Points in space will be designated by the small letters, as a, b, c. The V projections will be designated by the same letters with the exponent v, as a^v, b^v, c^v, and the H projections by the same letters with the exponent h, as a^h, b^h, c^h. Successive positions of the same points will be designated by subscripts, as a_1, a_2, a_3.

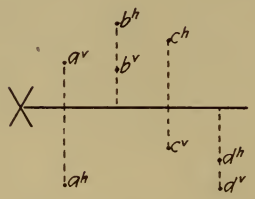


FIG. 4.

Problem 1.—*Having the direction and distance of a point in space from H and V, to draw its projections.*

Draw any line perpendicular to X and set off from X, above if the point is above H, or below if the point is below H, the distance of the point from H; this will be the V projection of the point. On the same perpendicular set off from X, above if the point is in back of V, or below if the point is in front of V, the distance of the point from V; this will be the H projection of the point.

Problem 2.—*Having one projection of a point in space and the direction and distance of the point from that plane of projection, to draw the other projection.*

A perpendicular to X through the given projection will contain the other projection, which will then be located as in Problem 1.

Problem 3.—*Having the projections of a point in space, to determine its position with reference to H and V.*

The distance of the H projection from X will show the distance of the point from V, and if the H projection is above X the point is back of V; if below X, in front of V.

In like manner the distance of the V projection from X will show the distance of the point from H. If the V projection is above X the point is above H; if below, the point is below H.

Solution of Exercises.—All exercises are to be drawn full scale unless otherwise noted. The exercises are designed to fit on standard letter size paper, 8½ in. × 11 in. The H and V projections of a point should always be connected by a projecting line perpendicular to X, preferably a dotted line as shown in the figures.

The universal use of the third quadrant in practical work makes it desirable to use this quadrant in the solution

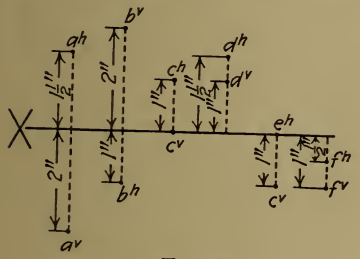


Fig. 5.

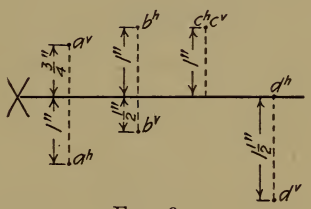


Fig. 6.

of exercises. In a few of the exercises the data specified definitely locate the object in one of the other quadrants, but wherever any choice is left, the third quadrant should be used.

Exercises

1. Show the projections of the following points:
   - *a*, 1 in. behind V, 1½ in. below H.
   - *b*, 2 in. behind V, 1 in. above H.
   - *c*, 3 in. in front of V, 1 in. above H.
   - *d*, 1 in. in front of V, 1 in. below H.
   - *e*, in V, 1 in. below H.
   - *f*, in V, 2 in. above H.
   - *g*, 1 in. behind V, in H.
   - *k*, 1 in. in front of V, in H.
   - *l*, in V, in H.

2. State where each of the points in Fig. 5 is located, giving distances and directions from H and V.

3. Show the projections of four points, one in each quadrant, each 1 in. from H and 2 in. from V.

4. State in which quadrant each of the points shown in Fig. 6 is located, and whether the point is nearer V or H.

5. The plan of a point is 1 in. above X, and the point is 1¼ in. below H. Show its projections.

6. A point 1 in. above X is the plan of three points, *a*, *b* and *c*. *a* is 1 in. above H, *b* is in H and *c* is 1½ in. below H. Show the projections of the points.

7. Three points, *a*, *b* and *c*, lie in a plane perpendicular to X. *a* and *b* have their elevations in the same point, and *b* and *c* have their plans in the same point. *b* is 2 in. from H and V, *a* is ½ in. from V and *c* is 1 in. from H. Show the projections of the points.

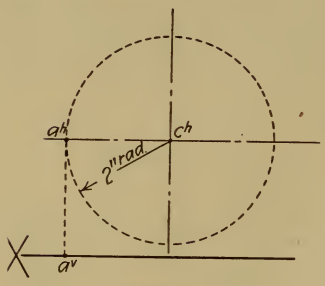


Fig. 7.

8. The elevations of two points coincide at a point 1½ in. below X. The points are equidistant from H and V, one in front of V and the other behind V. Show the projections of the points.

9. A point in the third quadrant which had its plan ½ in. from X and its elevation 2 in. from X has moved 2 in. to the right and ½ in. down. Show the projections of its former and present positions.

10. A point in the first quadrant is 1 in. from H and 2 in. from V. It moves ½ in. further from H, ½ in. nearer V and 1 in. to the right. Show the projections of its original and final positions.

11. The point *a* in Fig. 7 moves about a vertical line through *c* in such a way that the plan of its path is a circle. For each 30° that a^h moves about c^h, *a* moves ⅛ in. away from H. Show the projections of the path of the point for one complete revolution.

III. Representation of Lines.

The projecting lines of all the points of a right line in space form its projecting plane, the intersection of which with the plane of projection determines the projection of the line in space. The projection of a line in space is therefore determined by joining the corresponding projections

of any two points of the line. Hence, since any two points in the same perpendicular to X may be taken as the projections of a point in space, it will be seen that any two lines may be taken as the projections of a line in space, provided that they can be cut by two perpendiculars to X.

NOTATION.—Lines in space will be designated by capital letters, as A, B, C. The V projections of the lines will be designated by the same letters with the exponent v, as A^v, B^v, C^v, and the H projections by the same letters with the exponent h, as A^h, B^h, C^h. A line determined by the points a and b will be called the line $\overline{ab}$. Also the point determined by the intersection of the two lines C and D will be called the point $\overline{CD}$.

The points of intersection of a line with the H and V planes of projection are called its H and V traces, and are indicated by h and v.

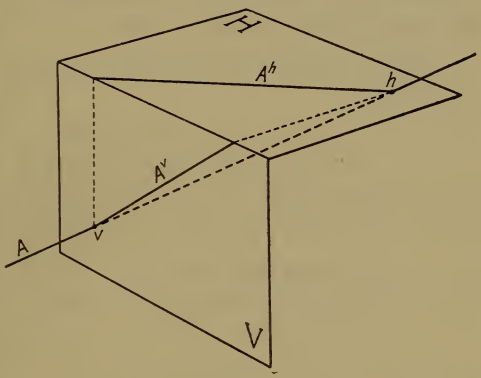


FIG. 8.

Problem 4.—*Having the projections of a line, to find its traces.*

The H trace of a line is a point in the H plane, therefore its V projection must lie in X. Since the H trace is a point of the line, its V projection must lie in the V projection of the line. If, therefore, the V projection of the line is extended till it cuts X, this point will be the V projection of the H trace. A perpendicular to X at this point will cross the H projection of the line at the H trace. Similarly, to find the V trace extend the H projection till it cuts X, then a perpendicular to X at this point will cross the V projection of the line at the V trace (Fig. 8).

NOTE.—A line is said to cross a given quadrant when the portion between the traces lies in that quadrant.

Since the projections of points determine the projections of lines containing them, exercises in the projection of lines must depend in general on Problems 1 and 2 to determine two points of the line, which is then itself determined.

A line parallel to H has all its points at the same distance from H. Therefore the V projection of the line must have all its points the same distance from X, or in other words must be parallel to X. Similarly a line parallel to V has its H projection parallel to X.

With reference to H and V a line may have six general positions:

Line	
A, parallel to H	parallel to V
B, parallel to H	perpendicular to V
C, parallel to H	inclined to V
D, perpendicular to H	parallel to V
E, inclined to H	parallel to V
F, inclined to H	inclined to V

H projection	V projection
A^h, parallel to X	A^v, parallel to X
B^h, perpendicular to X	B^v, a point
C^h, inclined to X	C^v, parallel to X
D^h, a point	D^v, perpendicular to X
E^h, parallel to X	E^v, inclined to X
F^h, inclined to X	F^v, inclined to X

Note an exception to F when the line is in a plane perpendicular to X. The projections then coincide in a line perpendicular to X and the line in space is indeterminate.

Exercises

12. Represent a line in each of the general positions indicated above.

13. Show the projections and traces of four lines, one crossing each

quadrant, the H and V projections of each line meeting X in points 2 in. apart.

14. Show the projections of a line 2 in. long lying in the fourth quadrant and perpendicular to H. One end of the line lies in H at a point 1 in. from V.

15. Show the projections of a line parallel to X, 1 in. from H and 2 in. from V.

16. A line crossing the second quadrant has its H and V traces each 2 in. from X. Its projections meet X at points 4 in. apart. Show the projections of the line.

17. Lines A and B meet at a point 1 in. from H and 2 in. from V. A is parallel to H and inclined at 45° to V. B is parallel to V and inclined at 30° to H. Show the projections and traces of the lines.

18. Show the projections and traces of a line passing through X and inclined to H and V.

19. Show the projections and traces of a line inclined at 30° to V, parallel to H and 1 in. from it.

20. The H trace of a line is 2 in. below X, the V trace 3 in. to the right and 1 in. below X. Show the projections of the line.

21. Show the projections and traces of the line containing the points a and b of Exercise 1, the projecting lines of the points being 1 in. apart.

22. Show the projections and traces of the line containing the points c and d of Exercise 1, the projecting lines of the points being 2 in. apart.

23. Show the projections and traces of the line containing the points b and c of Exercise 1, the projecting lines of the points being 2 in. apart.

24. Show the projections of the triangle having as vertices the points c, d and e of Exercise 1, the projecting lines of c and d coinciding and 1 in. from the projecting line of e.

25. Show the projections of the triangle having as vertices the points a, b and c of Exercise 2, the projecting lines of b and c being respectively 1 in. and 2½ in. from that of a.

26. Show the projections of all the triangles that may be formed having as vertices the points a, b, c and d of Exercise 1, the distances of the projecting lines of a to those of b, c and d being respectively 1 in., 2 in. and 3½ in.

27. The point $\overline{AB}$ is 1 in. from H and 1¾ in. from V; the H projection of this point and the H traces of the lines A and B form an equilateral triangle of 1½ in. side, of which one side is inclined at 45° to X. Find the V traces of A and B.

28. The point a is at the H trace and the point b at the V trace of the line $\overline{ab}$. The elevation of the line is inclined at 30° and the plan at 45° to X. The distance $a^v b^v$ is 2 in. Show the projections of the line.

IV. Representation of Planes.

Planes are generally represented by their lines of intersection with H and V, called H and V traces, forming

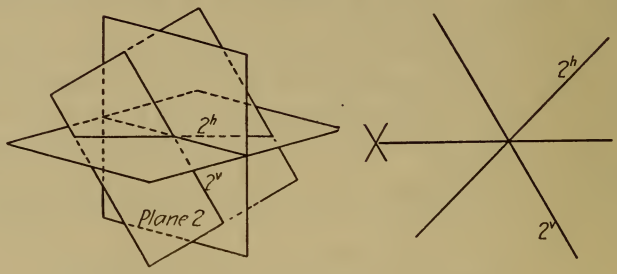


Fig. 9.

two lines intersecting in X and extending indefinitely on either side of it (Fig. 9).

For the sake of clearness it is customary to represent only the part of the plane included in one quadrant. The plane shown in Fig. 9 can be represented in four different ways, according as the part of the plane assumed is taken in the first, second, third or fourth quadrant (Fig. 10).

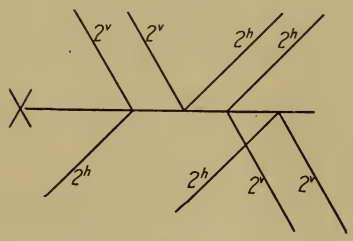


Fig. 10.

In the following, unless otherwise stated, planes will be taken in the third quadrant. The H trace is therefore represented above and the V trace below X.

Notation.—Planes, excepting the planes of projection, will be designated by the numerals, as *1*, *2*, *3*; the V trace by the same numeral with the exponent v, as 1^v, 2^v, 3^v; the H trace by the same

numeral with the exponent h, as 1^h, 2^h, 3^h. The plane determined by the points a, b and c will be called the plane $\overline{abc}$, and its traces $\overline{abc}^h$ and $\overline{abc}^v$. The plane determined by two intersecting lines A and B will be called the plane $\overline{AB}$, and its traces $\overline{AB}^h$ and $\overline{AB}^v$. The plane determined by the line A and the point b will be called the plane $\overline{Ab}$ and the traces $\overline{Ab}^h$ and $\overline{Ab}^v$. The line determined by the intersection of the planes *1* and *2* will be called the line $\overline{12}$, and the projections $\overline{12}^h$ and $\overline{12}^v$. The point determined by the intersection of the planes *1*, *2* and *3* will be called the point $\overline{123}$, and the projections $\overline{123}^h$ and $\overline{123}^v$. The point determined by the intersection of the plane 1 and the line A will be called the point $\overline{1A}$ and the projections $\overline{1A}^h$ and $\overline{1A}^v$.

With reference to H and V a plane may have eight general positions:

Plane	
1, perpendicular to H	perpendicular to V
2, inclined to H	perpendicular to V
3, perpendicular to H	inclined to V
4, inclined to H	inclined to V
5, perpendicular to H	parallel to V
6, parallel to H	perpendicular to V
7, parallel to, but not intersecting X	
8, passing through X	

H trace	V trace
1^h, perpendicular to X	1^v, perpendicular to X
2^h, perpendicular to X	2^v, inclined to X
3^h, inclined to X	3^v, perpendicular to X
4^h, inclined to X	4^v, inclined to X
5^h, parallel to X	5^v, at infinity
6^h, at infinity	6^v, parallel to X
7^h, parallel to X	7^v, parallel to X
8^h, in X	8^v, in X

Exercises

29. Represent a plane in each of the general positions indicated above.

30. Show the traces of a plane which is perpendicular to V and inclined at 30° to H.

31. Inclined at 45° to V and perpendicular to H.

32. Parallel to V and 1 in. behind it.

33. Parallel to H and ½ in. below it.

34. Parallel to H and 2 in. above it.

35. Parallel to V and ½ in. in front of it.

36. Represent the plane of Exercise 30 in the first quadrant.

37. Show the traces of a plane parallel to X and meeting H and V at an angle of 45°.

V. Representation of Simple Solids.

Solids are represented by showing the form and relative position of the surfaces bounding them. These surfaces are in turn known when we know their determining lines. Hence solids bounded by plane surfaces are represented by the lines of intersection of these surfaces. Solids bounded by curved surfaces are generally represented by their apparent boundary lines. The solids may be placed at any convenient distances from the planes of projection. Hidden lines are represented by short dashed lines.

Exercises

38. Draw the plan and elevation of a rectangular prism having the base parallel to H and a side face parallel to V, the base being 1 in. × 3 in. and the altitude 2 in.

39. Draw the projections of a cube of 2 in. edge with top and bottom faces parallel to H and a diagonal parallel to V.

40. Show the projections of a triangular prism with its axis vertical and a side face making an angle of 30° with V. Its base is equilateral of 2 in. side and its altitude 3 in.

41. Draw the projections of a cylinder 2 in. long and 2 in. in diameter, axis perpendicular to V.

42. Show the plan and elevation of a cone of revolution having for its base a 3 in. circle in H, while its vertex is 2 in. below H.

43. Draw the projections of a pyramid, the base being a square of 2 in. side situated parallel to V and 3 in. from it. The vertex is in V.

44. Draw the projections of a 2 in. sphere whose center is ½ in. from H and 2 in. from V.

45. Draw the projections of a cone of revolution whose base is 2 in. in diameter and altitude 3 in. Its axis lies in V and its base in H.

46. A sphere of 2 in. diameter has a vertical hole 1 in. in diameter bored through its center. Show its plan and elevation.

47. A torus, or anchor ring, is generated by revolving a 1 in. circle about a vertical axis distant 1 in. from the center of the circle. Show H and V projections.

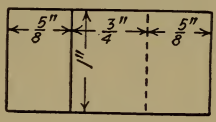


Fig. 11.

48. Figure 11 shows the plan of a stick whose right section is 1 inch square. Show the elevation.

49. A cylinder 2 in. in diameter and 3 in. long has its axis parallel to X. Show its projections. Note that the projections do not fully define the shape of the cylinder.

VI. Assuming New Planes of Projection.

As has been noted in the representation of lines, when the line is in a plane perpendicular to X it is indeterminate in direction or position, and may be straight or curved. Hence it is evident that in the representation of objects bounded by plane surfaces, when any of these planes comes into a position perpendicular to X, the outline of the surface cannot be determined from the plan and elevation. Therefore it is frequently necessary to show more than two views of a given object.

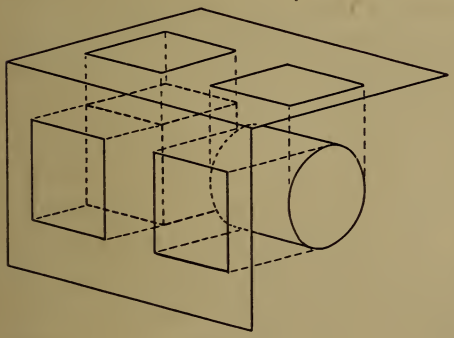


Fig. 12.

As an example, the cylinder of Exercise 49 is represented by two equal rectangles, of which the lines parallel to X will represent the sides of the cylinder

and the lines perpendicular to X the ends of the cylinder. But these two views would represent as well a prism of square section, Fig. 12, so that to determine the form fully an end view is necessary. The new plane of projection is usually taken perpendicular to both H and V, and is then called the Profile plane and is denoted by P. The line of intersection of P and V is called Y, and the line of intersection of P and H is called Z. P projections of points and lines, and P traces of planes will be indicated by the exponent p, as a^p, B^p and 3^p.

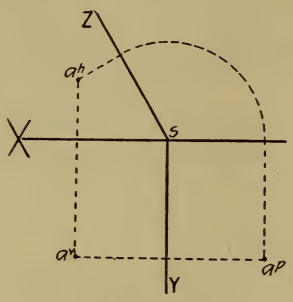


Fig. 13.

In some cases it is more convenient to take the third plane of projection perpendicular to H but inclined to V, as for example when it is desired to get a true end view of a solid whose end face is inclined to V. The general method of finding the third projection is the same in either case.

Problem 5.—*Given the H and V projections of a point in space, to show its projection on a new vertical plane.*

To bring the new vertical plane into the plane of the paper it is customary to revolve it about its V trace or Y. This revolution is made in such a direction that Z moves clockwise into coincidence with X. The P projection of the point will fall the same distance below or above X as the V projection. To find the distance of the P projection to the right or left of the Y axis, draw a perpendicular from a^h to Z (Fig. 13). The required distance will evidently be the distance between the foot of the perpendicular and the point s, and this distance can be conveniently laid off by drawing the arc about s as center, as shown in the figure.

If the new plane of projection is perpendicular to both H and V, Y and Z form one straight line perpendicular to X,

and the perpendicular from a^h to Z becomes a horizontal line. This is illustrated in Fig. 14, which shows the projections of a triangular prism.

Exercises

50. Show H, V and P projections of the cylinder of Exercise 49.

51. Show the P projections of the points *a*, *b* and *c* of Exercise 1.

52. Show the P projection of the line $\overline{ab}$ of Exercise 21.

53. Show the P projection of the triangle *cde* of Exercise 24.

54. Points *a* and *b* lie in a plane perpendicular to X. *a* is 1½ in. from H and ½ in. from V. *b* is 1 in. from H and 1½ in. from V. Show projections and traces of the line $\overline{ab}$.

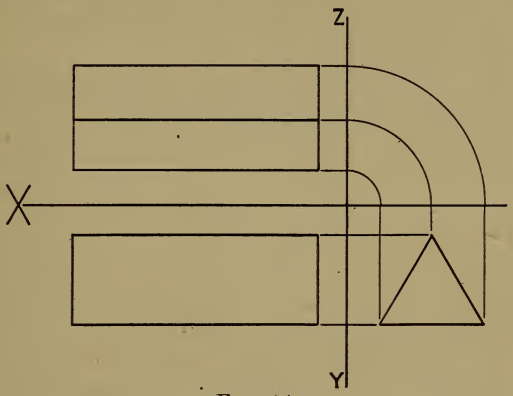


Fig. 14.

55. Point *c* lies 1 in. from V, 2 in. from H and 3 in. from P. Point *d* lies 2 in. from V, 1 in. from H and 1 in. from P. Find H, V and P traces of the line $\overline{cd}$.

56. The H trace of a line is 2 in. above X and the V trace 1 in. below X, both traces being in the same perpendicular to X. Show H, V and P projections of the line.

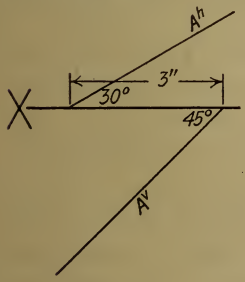


Fig. 15.

57. Plane 2 has its H trace inclined at 30° and its V trace at 45° to X. Find its P trace.

58. Find the projections of a point of line *A* which is twice as far from H as it is from V, Fig. 15.

59. Show the traces of a plane parallel to X and inclined at 30° to H.

60. Draw the plan and elevation of a bolt 1⅛ in. diameter, 3 in. long under the head, the head being hexagonal, 1¾ in. short diameter and 1 in. thick. Take axis parallel to X.

61. Show the projections of a square frame 4 in. outside, the sides

of the frame being 1 in. square. Take a 4 in. × 4 in. face parallel to V.

62. Show the projections of the frame of the preceding exercise when the long dimension of the plan is inclined at 30° to X.

63. Show the projections of a hexagonal prism 2 in. long and 2 in. long diameter, with the axis parallel to H and inclined at 30° to V.

64. Show top, side and end views of a bolt having a hemispherical head of 2 in. diameter. The shank for 1½ in. under the head is 1 in. square, the remaining 2 in. cylindrical and 1 in. in diameter. Take axis parallel to X.

65. Project a circle of 2 in. diameter lying in a plane parallel to V, on a plane perpendicular to H and inclined at 60° to V.

66. Project a cube of 1½ in. edge on a plane perpendicular to one

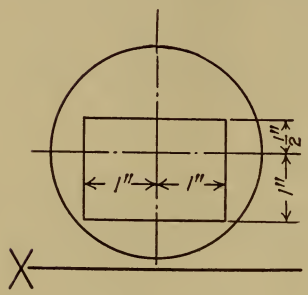


Fig. 16.

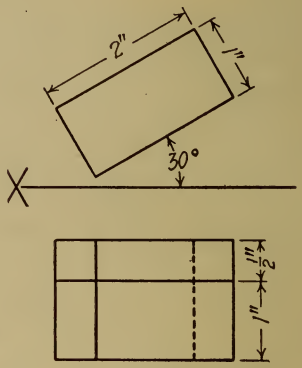


Fig. 17.

of its diagonals. Assume the position of the cube so that one of its diagonals is parallel to H.

67. A cone has for its elevation an equilateral triangle of 2 in. side. The vertex of the cone is in H and the plane of the base makes an angle of 45° with H. Show plan, front and end elevations.

68. A sphere of 2½ in. diameter is pierced by a vertical hole 1½ in. square. Show top and front view.

69. In Fig. 16 is shown the H projection of a 3 in. sphere pierced by a vertical rectangular hole. Show V and P projections.

70. A hexagonal prism of 3 in. length, 2 in. long diameter, has a cylindrical hole of ¾ in. diameter half its length. Show plan and elevations with axis parallel to X.

71. Show the projections of the box of Fig. 17 with the cover raised at an angle of 45°.

72. A rectangular block $1\frac{1}{2}$ in. $\times$ 2 in. $\times$ 3 in. has two of its 3 in. edges in planes $2\frac{1}{4}$ in. apart and parallel to H. Show plan and elevation of the block.

73. A set of four equal steps having each a rise and tread of 8 in. is 4 ft. long. Show elevation when the long dimension of the plan is inclined at 30° to X. Scale, $\frac{3}{4}$ in. = 1 ft. 0 in.

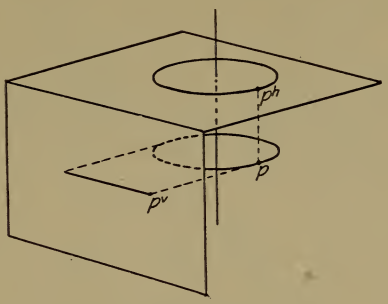


Fig. 18.

VII. Change of Position by Rotation.

The views obtained by assuming new planes of projection can also be obtained by changing the position of the object with reference to H and V by rotation about some axis. When a point is revolved about an axis, it describes the circumference of a circle, the plane of the circle is perpendicular to the axis, its center is in the axis and its radius is the shortest distance from the point to the axis. If the axis is oblique to the planes of projection the path of the point will be projected as ellipses. For ease and simplicity of construction the axis should be taken perpendicular to one of the planes of projection. The result obtained by rotating about an oblique axis can be obtained by two rotations about perpendicular axes, one perpendicular to V, the other to H.

If the axis is taken perpendicular to H, then the H projection of the path of any point will be a circular arc with center at the H projection of the axis, and the V projection of the path a straight line parallel to X and through the V projection of the point (Fig. 18). Correspondingly, if the axis is taken perpendicular to V, the V projection of the path of any point is an arc of a circle and the H projection a straight line parallel to X.

Figure 19 shows the rotation of a line B about an axis A, the axis being perpendicular to V. This problem would be greatly simplified by taking the axis of rotation intersecting the line B, in which case the point of intersection would be a stationary point.

Figure 20 shows a solution of Exercise 66 by rotating the cube so that a diagonal is perpendicular to V, in which case the new V projection is the required projection.

There are certain cases in which the limitations given

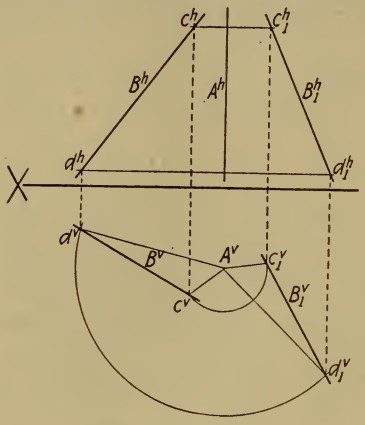


FIG. 19.

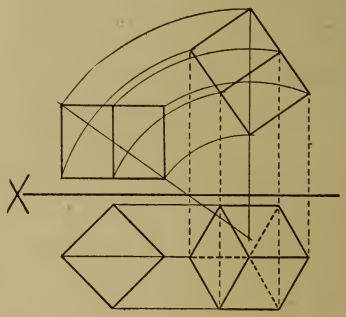


FIG. 20.

above as to the positions of the axes are not necessary. (*a*) The revolution of a point into either plane of projection about any line in that plane as axis. (*b*) The revolution of a line into either plane of projection about an axis lying in that plane and passing through the corresponding trace of the line. (*c*) The revolution of a plane into either plane of projection about its intersection with that plane as axis.

Problem 6.—*To revolve a given point into one of the planes of projection about an axis in that plane, the axis being inclined to X and containing the projection of the point.*

Given point p and the axis in H, to revolve the point

into H (Fig. 21). Since the plane of revolution of the point will be perpendicular to the axis of revolution, a line drawn through p^h perpendicular to the axis will contain the revolved position of the point. The radius of the circle described by the point is the shortest distance from the point to the axis and is measured by the distance from p^v to X. This distance set off from p^h along the perpendicular will locate the revolved position of p.

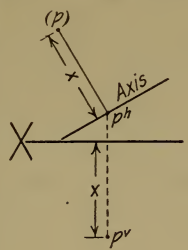


Fig. 21.

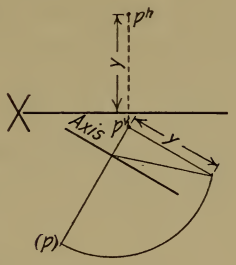


Fig. 22.

The position of a point after revolution into one of the planes of projection will be designated by enclosing the letter in parentheses, as (p).

Problem 7.—*To revolve a given point into one of the planes of projection about an axis in that plane, the axis being inclined to X and not containing the projection of the point.*

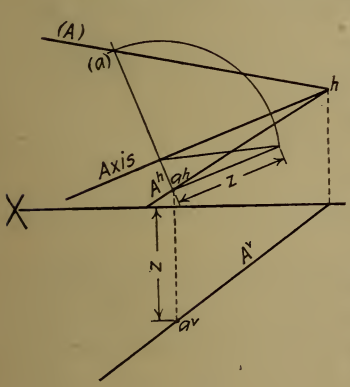


Fig. 23.

Given point p and the axis in V, to revolve the point into V (Fig. 22). In this case, as before, the revolved position of the point will lie in a perpendicular to the axis of revolution and passing through p^v. The radius of the circle is the shortest distance from the point to the axis, and is found as the hypotenuse of a right triangle having the distances from p^h to X and from p^v to the axis as sides. This distance being found by construction is set off from the axis, giving (p) as shown.

Problem 8.—*To revolve a given line into one of the planes of projection about an axis in that plane and passing through the corresponding trace of the line.*

Given line A and the axis in H passing through the H trace, to revolve the line into H (Fig. 23). It is evident that the H trace of the line, being the point of intersection of the line and the axis of revolution, will not change its position during the revolution. If any other point of the line is assumed, such as a, and its revolved position is found as in Problem 7, the revolved position of the line

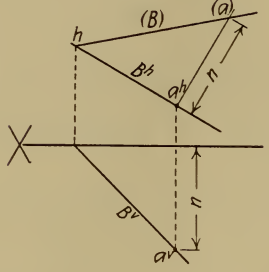


Fig. 24.

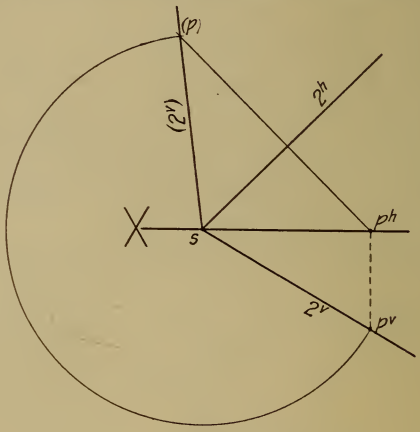


Fig. 25.

will be the line joining the revolved position of the point with the H trace of the line.

This problem is simplified by letting the axis of revolution coincide with the H projection of the line (Fig. 24). The revolution of the assumed point will then be accomplished by the method of Problem 6.

Note.—If the axis of revolution does not pass through the H trace of the line, that is, if the line and axis are not in the same plane, the problem is impossible.

Problem 9.—*To revolve a given plane into one of the planes of projection about its trace as axis.*

Given plane *2*, to revolve it into H about its H trace (Fig. 25). This result is effected by taking any point of the V trace and revolving it about the H trace into H. The position of (p) may be found by the right triangle construction of Problem 7, but a simpler method is to draw an arc with center at s and with the distance sp^v as radius. The point in which the arc cuts the perpendicular to 2^h from p^h is (p), since $s(p)$ and sp^v must be of equal length.

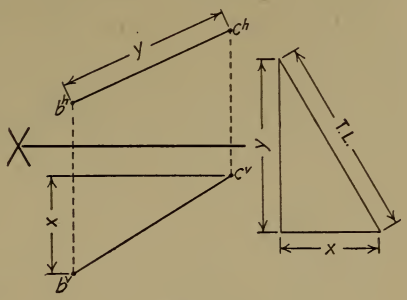


Fig. 26.

Problem 10.—*To find the true distance between two given points.*

First Method.—Revolve the line containing the points into or parallel to one of the planes of projection, whence the projection on that plane will show the true distance.

Second Method.—The true distance can be found as the hypotenuse of a right triangle, one side of which is the length of one projection of the distance, the other side the difference in distances of the two points from the corresponding plane of projection. In Fig. 26, let it be required to find the distance between b and c. Construct a right triangle with the length of b^hc^h as one side and the difference in distances from H, or the length x as the other side, then the hypotenuse is the required true distance.

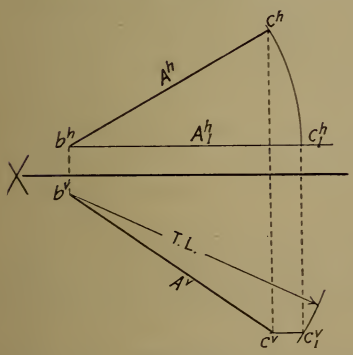


Fig. 27.

Problem 11.—*Given of a line one projection and its true length, to find the other projection.*

Given A^h and the true length of A (Fig. 27). Either method of Problem 10 can be applied.

First Method.—Assume the V projection of one point on the line, as b^v. Suppose the line to be revolved parallel to V about an axis through b, giving the new H projection as $A_1{}^h$. Then the V projection of the line will have one end at

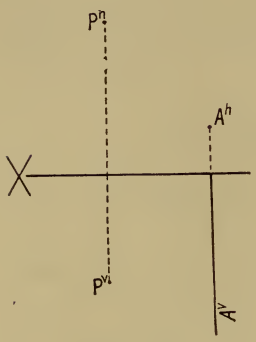


Fig. 28.

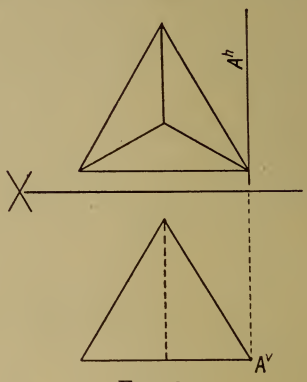


Fig. 29.

b^v, the other end in the perpendicular dropped from $c_1{}^h$, and will have a length equal to the given true length. Counter revolve the line to its original position, giving A^v.

Second Method.—Construct a right triangle with the length of the given H projection as one side and the true length as hypotenuse. The other side will be the difference in distances from H of the points b and c.

Exercises

74. Revolve p through 60° about A as axis (Fig. 28).

75. A regular tetrahedron of 3 in. edge has its base in a plane parallel to H and 3 in. from it, one side of the base being parallel to V. Show its projections.

76. Revolve the tetrahedron shown in Fig. 29 through 105° about A as axis.

77. Revolve B into a position parallel to V (Fig. 30).

78. Revolve C into a position parallel to X (Fig. 31).

79. Revolve *D* into a position perpendicular to V (Fig. 32).

80. A square of 2 in. side stands in a plane perpendicular to X, with its sides parallel to H and V. Revolve about one of the vertical sides through 60°, then about an axis perpendicular to V and containing a corner of the square, through 120°.

81. A regular hexagon of 1½ in. side lies in a plane perpendicular

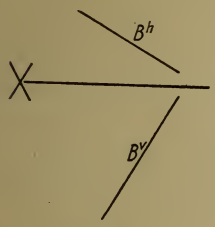


Fig. 30.

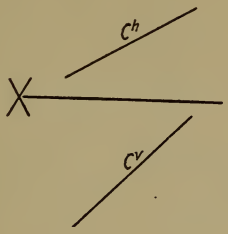


Fig. 31.

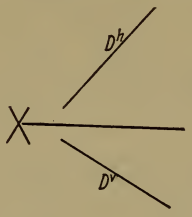


Fig. 32.

to X, with two sides horizontal. Revolve the hexagon through 45° about a vertical axis passing through a corner of the hexagon.

82. Point *a* is in H, 2 in. behind V; point *b* is 1 in. to the right, 1 in. below H and 1 in. behind V; point *c* is 1 in. to the right of *b*, 1½ in. below H and 2 in. behind V. Show projections and true form of the triangle *abc*.

83. A 3 in. circle lies in a plane parallel to V and 2 in. from it. The circle is revolved about the vertical diameter as axis until its V projection is an ellipse with minor axis 1¼ in. long. Show projections of the circle before and after rotation. Find at least twelve points of the revolved position.

84. The H trace of a line is 2 in. above X, the V trace 1½ in. below X and the line is 6 in. long between traces. Show its projections.

85. Triangle *def* has *fd* = 3 in., *fe* = 4 in. and *de* = 5 in. *d* and *e* lie in X and *f* is 1 in. below H. Show projections of the triangle.

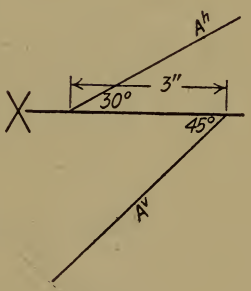


Fig. 33.

86. The H trace of line *A* is 2 in. from X. A^h is inclined at 30° to X and is 3 in. long. The true length of *A* is 4 in. Find the V projection and V trace of *A*.

87. Show the projections of a point on line *A* at a true distance of 1½ in. from the H trace (Fig. 33).

88. The P trace of a line is 1 in. from H and 2 in. from V. The V trace is ½ in. from H and 1½ in. from P. Find the true distance between the H and P traces.

89. A line 2 in. long between its H and P traces has its H trace 1 in. from V and 1¼ in. from P. Its P trace is 1 in. from H. Show its projections.

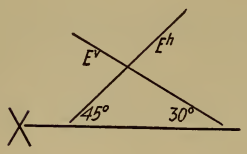


Fig. 34.

90. A rod 2½ in. long is suspended horizontally by vertical threads 3 in. long attached to its ends. Show how far the rod will be raised by turning it through 90°.

91. A rod 2 in. long stands perpendicular to H at a point 2 in. from V. A second rod 2 in. long stands perpendicular to V at a point 1½ in. from H. The distance from the foot of one rod to the foot of the other is 4 in. What is the distance between the other ends of the rods?

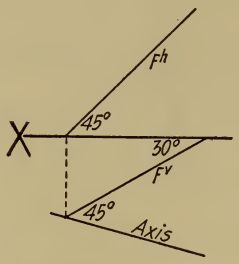


Fig. 35.

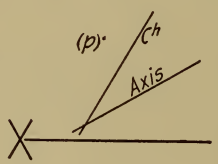


Fig. 36.

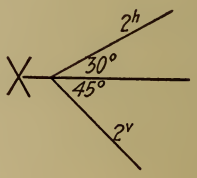


Fig. 37.

92. The plan of a triangle *abc* is equilateral of 2 in. side. $ab = 2\frac{1}{2}$ in., $bc = 3$ in. What is the length of ac?

93. Of the triangle *def*, $de = 1$ in., $ef = 1\frac{1}{2}$ in. and $df = 2$ in.

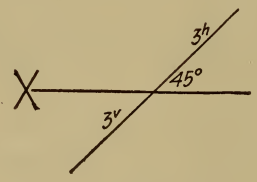


Fig. 38.

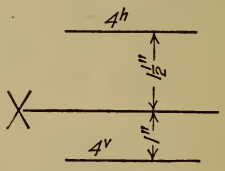


Fig. 39.

d and f are in X and e is equidistant between H and V. Show plan and elevation of the triangle.

94. Revolve E into H about E^h as axis (Fig. 34).

95. Revolve F into V about the axis shown in V (Fig. 35).

96. The line C, whose H projection is given, has been revolved into H about the axis, and (p) is a point on its revolved position. Find the V projection of C (Fig. 36).

97. Revolve the plane *2* into H (Fig. 37).

98. Revolve the plane *3* into V (Fig. 38).

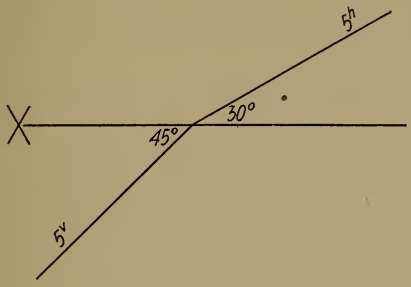


Fig. 40.

99. Revolve the plane *4* into H (Fig. 39).

100. Revolve the plane *5* into H (Fig. 40).

101. An isosceles triangle of 2 in. base and 3 in. sides lies in H with the base parallel to X. Revolve the triangle through 60° about one of the 3 in. sides.

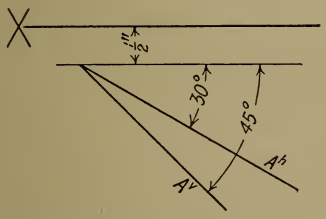


Fig. 41.

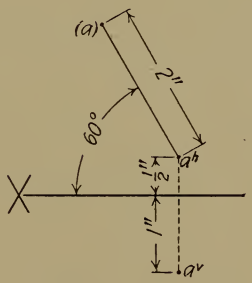


Fig. 42.

102. Revolve line A into H about A^h as axis (Fig. 41).

103. Point a, when revolved into H about an axis in H, moves to (a). Find the axis (Fig. 42).

VIII. Lines with Reference to H and V.

A line in space has the following distinctive data with reference to H and V: H projection, V projection, θ, the

angle of inclination with H, and ϕ, the angle of inclination with V. Any two of these are enough to determine the other two. The H and V traces of the line are important points, but are readily derived from the projections, or the projections from the traces, and in connection with other data have simply the effect of any other given points of the line.

Problem 12.—*Given the projections of a line, to find θ and ϕ (Fig. 43).*

First Method.—Since the angle between a line and a plane is the angle between the line and its projection on that plane, the angle θ is the angle between the line and its H projection. To find its value, revolve the line into H about its H projection, whence the angle between the revolved position of the line and its original H projection is the required angle. Similarly to find ϕ, revolve about the V projection into V.

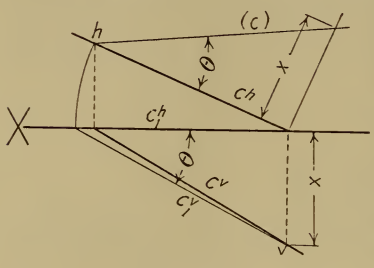


Fig. 43.

Second Method.—The angle between the V projection of the line and X is a projection of the angle θ. To find its true value revolve the line about an axis perpendicular to H until the H projection is parallel to X. The angle between the new V projection and X is the required angle. Similarly to find ϕ, revolve the line about an axis perpendicular to V until the V projection is parallel to X.

Figure 43 illustrates both methods of finding the angle θ.

Problem 13.—*Given of a line one projection and the angle which the line makes with the corresponding plane of projection, to find the other projection.*

Given A^h and θ (Fig. 44).

First Method.—From any point of A^h draw a line (A) making the required angle with it. This line will represent

the line in space revolved into H about A^h as axis, and the point chosen on A^h will represent the H trace of the line.

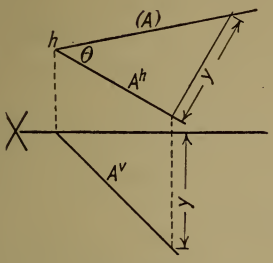


Fig. 44.

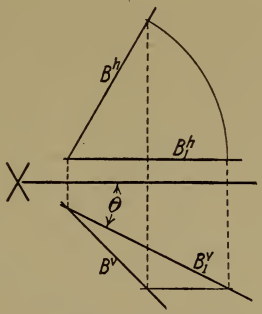


Fig. 45.

Make the counter revolution, which will give the required V projection.

Second Method.—Swing A^h parallel to X, draw the V projection making the given angle θ with X, and make the counter revolution.

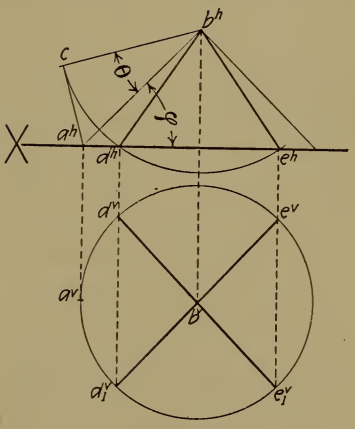


Fig. 46.

Problem 14.—*Given of a line one projection and the angle which the line makes with the other plane of projection, to find the other projection.*

Given B^v and θ (Fig. 45). Consider the line revolved into a position parallel to V, whence $B_1{}^v$ will make the angle θ with X and $B_1{}^h$ will be parallel to X. Make the counter revolution, which will give the required H projection. The counter revolution may be made in either direction, giving two possible positions.

Problem 15.—*To find the projections of a line making the angle θ with H and the angle ϕ with V.*

NOTE.—The limits of the value of $\theta + \phi$ are 0° and 90°.

First Method.—Suppose b in Fig. 46 to be one of the points of the required line, then all the lines passing through b and making an angle ϕ with V will lie on the surface of a cone having b as vertex and the circle $d^v e^v e_1{}^v d_1{}^v$ as a base, the line $b^h a^h$ representing the slant height of the cone and inclined at the angle ϕ to X. Construct the right triangle $b^h a^h c$ with the angle at b^h equal to the given angle θ. Then since $b^h a^h$ is the true length of every element of the cone, $b^h c$ will represent the length of the H projection of an element making the required angle with H. With this length as radius and b^h as center draw the arc cutting X at the points d^h and e^h. The corresponding V projections will be d^v, $d_1{}^v$, e^v and $e_1{}^v$, giving four possible positions of the line.

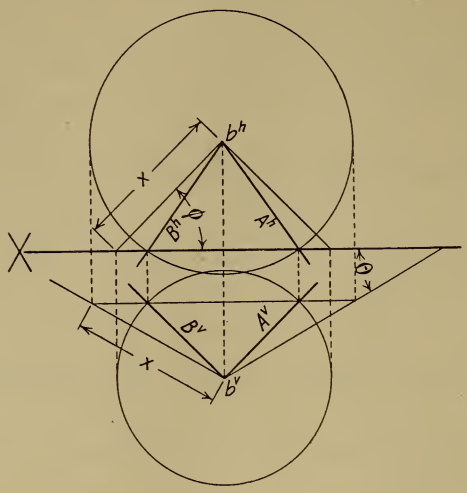


FIG. 47.

This problem may be solved equally well by drawing a θ cone with its vertex at b and base in H, and using the angle ϕ in the construction of the right triangle. In some cases it may be more convenient to place the base of the θ or ϕ cone parallel to instead of in H or V.

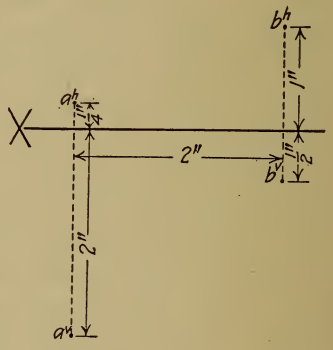


FIG. 48.

Second Method (Fig. 47).—Construct both θ and ϕ cones as above, then an element which lies on both cones will be the required line. In order to find such a line it is necessary that the slant heights of the two cones be made equal. In this case the bases of the cones will be intersecting circles and the line joining a point of intersection with the common vertex b will be a solution of the problem. In order to get all four possible solutions it is necessary to construct both nappes of one of the cones.

Exercises

104. Find θ and ϕ for the line $\overline{ab}$, Fig. 48.

105. A line having its H trace $1\frac{1}{2}$ in. from X meets V 2 in. below X. Find θ when ϕ is 30°.

106. A line has its H projection inclined at 30° to X. Its V trace is 2 in. below X. θ is 30°. Show projections and traces of the line.

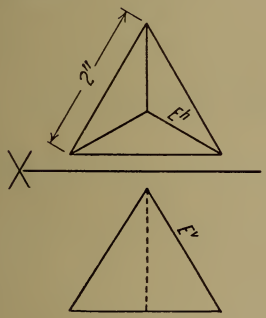


Fig. 49.

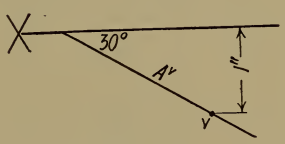


Fig. 50.

107. Triangle abc (ab = 2 in., ac = $1\frac{1}{2}$ in. and bc = $2\frac{1}{2}$ in.) lies in a plane perpendicular to H and inclined at 45° to V. The side bc is inclined at 30° to H. Show plan and elevation of the triangle.

108. A line making 60° with X is the plan of a line which makes 45° with its V projection. Show the V projection.

109. Figure 49 shows a regular tetrahedron of 2 in. edge. Find θ and ϕ for the edge E.

110. A line has its H trace 1 in. from X and its V trace 3 in. to the right and 2 in. from X. Find its true length and its angles θ and ϕ.

111. Find A^h when ϕ is 45° (Fig. 50).

112. A ray from a point 3 in. from H and 2 in. from V is reflected

from a point in V 1½ in. from H and 2 in. to the right. Where will the reflected ray meet H?

113. A triangle having sides 4 in., 3 in. and 2 in. lying in H with the 3 in. side in X, is revolved about the 4 in. side until the plane of the triangle is vertical. What angle does the 3 in. side then make with V?

114. In the orthographic projection of shadows the direction of the

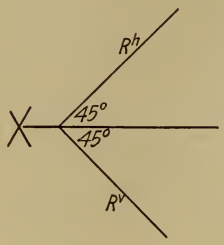


Fig. 51.

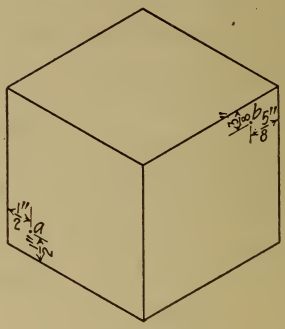


Fig. 52.

projections of the rays of light are assumed as shown in Fig. 51. What angle does a ray make with H and V?

115. A ray of light is inclined at 45° to V and its plan is inclined at 60° to X. Assume a point in the first quadrant 1 in. from V and ½ in. from H, and find its shadows on H and V.

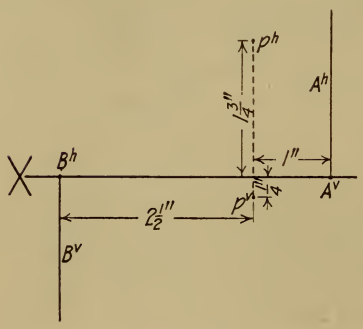


Fig. 53.

116. A cube of 3 in. edge has a hole drilled at *a* ½ in. deep and at *b* 1 in. deep. At what point of the top and at what angles with top and front faces must a drill be started to join the ends of the holes (Fig. 52)?

117. A sphere which is tangent to both H and V has its center on a line which meets H 2 in. from X and meets V 4 in. to the right and 3 in. from X. Show projections of the sphere.

118. A line drawn through p intersects the lines A and B (Fig. 53). Find θ and ϕ for the line.

119. A line is drawn from one corner to the center of a rectangular block 2 in. × 3½ in. × 5 in. At what angles is this line inclined to the three faces of the block?

120. The triangle *abc* (*ab* = 2 in., *bc* = 3 in. and *ac* = 3½ in.) has the point *a* in V, 2½ in. from H and 1½ in. from P. *b* is in P, 2 in. from H. *c* is in H. Show the projections of the triangle.

121. Point *d* lies in V, 2 in. from H and 1½ in. from P. Point *e* lies in H, 1 in. from V and 2 in. from P. Point *f* lies in P, 2½ in. from *d* and 3 in. from *e*. Show H, V and P projections of the triangle *def*.

122. A rectangle ½ in. × 2 in. has its short edges parallel to H. Its long edges are inclined at 30° to H and at 30° to V. Show H and V projections of the rectangle.

123. A line 3 in. long has one end in H and the other in V. $\theta = 15°$, $\phi = 45$. Show the projections of the line.

124. A pyramid having an altitude of 1½ in. has its axis in X and the sides of the base parallel to H and V. The face edges of the pyramid make 30° with H and 45° with V. Show a P projection of the pyramid.

125. A rectangular block 2 in. × 3 in. × 4 in. has a hole drilled through its center, the axis of the hole being inclined at 45° to a 3 in. × 4 in. face and at 30° to a 2 in. × 4 in. face. Show where the drill pierces the faces of the block.

126. A line 3 in. long has one end in H. The other end lies at a point 2 in. from H and 3 in. from V. $\phi = 30°$. Show projections of the line and find its angle θ.

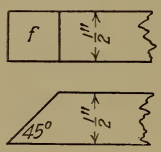


Fig. 54.

127. The stick shown incomplete in Fig. 54 has the end face *f* against H. The other end face is cut to fit against P. The stick is inclined at 30° to V. Complete the views indicated.

128. Show the projections of a line inclined at 30° to H and at 60° to V.

IX. Planes with Reference to H and V.

A plane in space has the following distinctive data with reference to H and V (Fig. 55). its H trace, its V trace, the angle *K*, or true angle between the H and V traces, and the angles θ and ϕ, or the angles made by the plane with H and V respectively. Any two of these are enough to determine the other three.

Problem 16.—*Given the traces of a plane, to find the angles of inclination with H and V* (*Fig.* 56).

If a plane is tangent to a cone of revolution, the plane

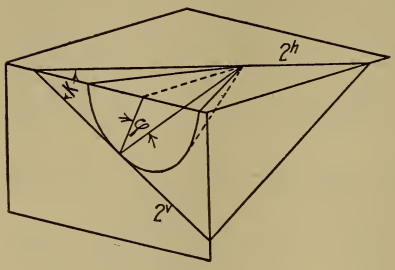


FIG. 55.

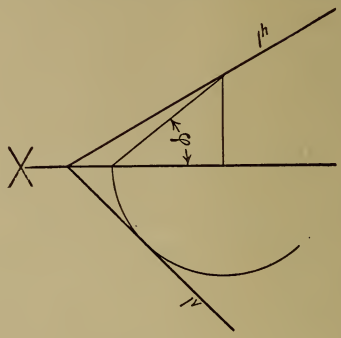


FIG. 56.

will make the same angle with the plane of the base that an element does. Hence, to find ϕ construct a cone with base in V and tangent to the V trace of the plane, and its axis in H and vertex in the H trace of the plane. The angle which the elements of this cone make with V will be the required angle. Similarly to find θ take the base of the cone in H tangent to the H trace, and the axis in V, vertex in the V trace.

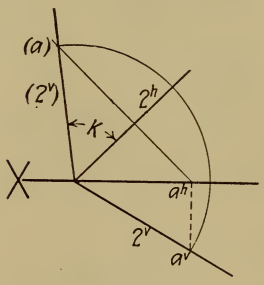


FIG. 57.

Problem 17.—*Given the traces of a plane, to find the angle K between the traces* (*Fig.* 57).

Revolve the plane into H or V, as in Problem 9, and the true angle will appear.

Problem 18.—*Given one trace of a plane and the angle which the plane makes with the corresponding plane of projection, to find the other trace.*

Given 3^h and θ (Fig. 58). Assume the base of a cone in H and tangent to 3^h, the axis of the cone being in V.

Draw an element of the cone making the given angle θ with H, thus locating the vertex, through which 3^v must pass. There are two solutions of this problem, according as the vertex of the cone is taken above or below X.

Problem 19.—*Given one trace of a plane and the angle which the plane makes with the other plane of projection, to find the other trace.*

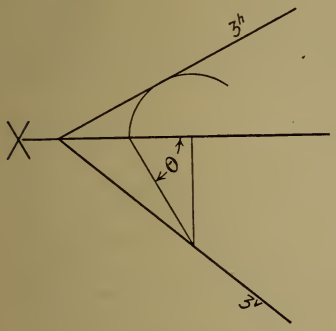


Fig. 58.

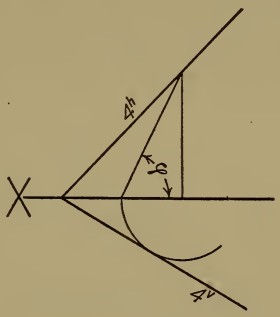


Fig. 59.

Given 4^h and ϕ (Fig. 59). From any point of 4^h draw a line perpendicular to X, which will represent the axis of a cone having its base in V. From the same point of 4^h draw a line making the given angle ϕ with X. This will represent an element of the same cone. Now draw the base of the cone, and 4^v will be tangent to it as shown. There are two solutions, according as 4^v is tangent above or below X.

Problem 20.—*Given one trace of a plane and the angle K, to draw the other trace.*

Given 2^h and K (Fig. 57). Lay off the given angle as though the plane had been revolved into H. Make the counter revolution, which gives the required trace.

Problem 21.—*Of a plane given the angles K and θ or ϕ, to draw the traces.*

Given K and θ (Fig. 60). Construct a cone with base in

H, axis in V, the elements of which make the given angle θ with H. If the required plane is tangent to this cone, the element of tangency and the traces form a right triangle of which a side and two angles are known. Construct a right triangle with an element of the cone as one side, and the given angle K as the opposite angle. If this triangle is conceived to move in such a way that A remains on the surface of the cone and B remains in V, then when the point

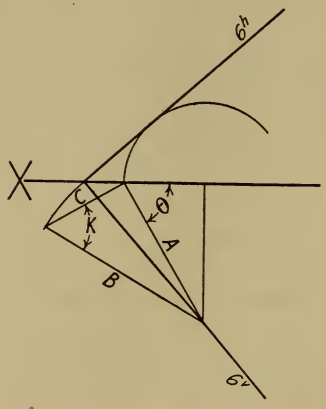


Fig. 60.

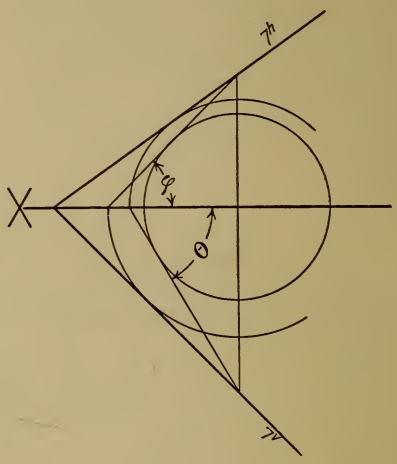


Fig. 61.

$\overline{CB}$ comes to X, C and B will coincide with the traces of the plane. Take the vertex of the cone as center and with the length of B as radius draw an arc. From the point where this arc cuts X draw the H trace tangent to the base of the cone, and the V trace through the vertex of the cone.

Problem 22.—*Given the angles θ and ϕ, to draw the traces of the plane.*

Note.—The limits of the value of $\theta + \phi$ are 90° and 180°.

A plane tangent to a θ cone and also to a ϕ cone will evidently make the required angles with H and V. In order to locate the cones in such a way that it will be pos-

sible to draw one plane tangent to both of them, it is necessary that both cones be tangent to a sphere with its center in X. With any convenient radius draw the projections of the sphere, Fig. 61, and draw the θ and ϕ cones tangent to it. The H trace of the plane will pass through the vertex of the ϕ cone tangent to the base of the θ cone. The V trace will pass through the vertex of the θ cone tangent to the base of the ϕ cone.

Exercises

129. 2^h is perpendicular and 2^v is inclined at 60° to X. Find θ and ϕ.

130. 3^h and 3^v appear as one straight line making 60° with X. Find θ and ϕ.

131. K equals 75°, θ equals 60°. Find the traces of the plane.

132. θ equals 75°, ϕ equals 30°. Find the traces of the plane.

133. The H trace of a plane is inclined at 30° to X. K is 60°. Find the V trace of the plane and its angles θ and ϕ.

134. θ equals 30°, ϕ equals 60°. Find the traces of the plane.

135. ϕ equals 45°, K equals 90°. Find the traces of the plane.

136. ϕ equals 45°, K equals 60°. Find the traces of the plane.

137. 3^h is inclined at 30° to X. ϕ equals 60°. Find θ.

138. 4^h is inclined at 15° to X. K equals 120°. Find θ and ϕ.

139. A corner of a cube is cut off in such a way that the face left makes 135° with one face of the cube and 120° with the second. What angle does it make with the third face?

140. The H, V and P traces of a plane form a triangle of sides 2 in., 2½ in. and 3 in. What angle does the plane make with H, V and P?

141. The oblique section of a square stick makes angles of 75° and 45° with adjacent faces. The longer sides of the section measure 1½ in. Show a right section of the stick.

142. On a square building having a hip roof the hip rafter makes an angle of 30° with H. What angle does the plane of the roof make with H?

143. To what angle must the top of the hip rafter be beveled to lie in the planes of the roof?

144. A building 20 ft. by 30 ft. has a hip roof, and the hip rafters are 18 ft. long. At what angles are the roof planes and the hip rafters inclined to the horizontal? Scale, ⅛ in. 1 ft. 0 in.

145. Triangle abc (ab = 2 in., bc = 3 in. and ac = 4 in.) lies in a

plane inclined at 45° to V. *ab* lies in H and *ac* in V. Show projections of the triangle.

146. Figure 62 shows the bottom portion of a bay window. A side face *abc* measures *ab* = 2½ ft., *ac* = 5 ft. and *bc* = 6 ft., and is inclined at 45° to H. Show the true form of the middle face. Scale, ½ in. = 1 ft. 0 in.

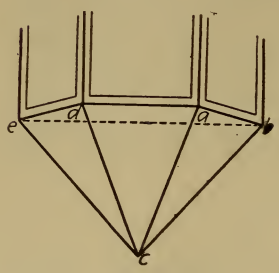


Fig. 62.

147. The edges *ac* and *dc* of the bottom portion of a bay window are inclined at 30° to the wall. *ed* and *ab* measure 2 ft., *da* 3 ft., and *eb* 5 ft. At what angle is the side face *abc* inclined to the wall (Fig. 62)? Scale, 1 in. = 1 ft. 0 in.

X. Development of Surfaces.

Developing a surface consists of unrolling or unfolding it until it lies entirely in one plane. The development of a surface is a pattern which may be cut out of paper or sheet metal and be rolled or folded so as to form the surface.

Whether or not a surface can be developed depends on the nature of the surface and the way it is generated.

When a line changes its position it generates a surface. The nature of this surface depends upon the kind of line and the character of its motion. The same surface may be generated in more than one way, as for example the cylinder of revolution which may be formed by revolving one of two parallel straight lines about the other as axis; it may also be generated by moving a circle so that its center travels along a straight line perpendicular to its plane.

A surface which can be generated by a straight line is called a ruled surface. Only ruled surfaces are developable, although approximate developments can be made of other surfaces. Not all ruled surfaces can be developed. Only those ruled surfaces can be developed which may be brought into complete coincidence with a plane without changing the relative positions of any two consecutive straight line

elements. This means that any two consecutive elements must be in the same plane, that is they must intersect or be parallel, as in the case of the cone or cylinder.

The line of shortest distance between two points on a developable surface will form a straight line in the developed surface.

Problem 23.—*To develop the surface of a polyhedron.*

The method of development must in general consist in finding the true shape and size of each of the faces, and arranging them so that as far as desirable the same edges will be coincident after development as before.

Problem 24.—*To develop the lateral surface of a prism.*

Consider the face of the prism as placed against a plane and the prism rolled about each lateral edge, unwrapping its faces as it turns, till each face has come in contact with the plane. The lateral edges of the prism will remain as parallel lines of distance apart equal to the breadth of the corresponding faces. The total width of the development will be the perimeter of the prism. Any right section of the prism will develop into a right line perpendicular to the edges. If the bases of the prism are oblique the lengths of the edges can be set off from the development of any assumed right section.

Problem 25.—*To develop the lateral surface of a pyramid.*

If the pyramid is rolled about its edges, as in the case of the prism, the vertex of the pyramid remains at a point and the edges develop as lines radiating from this point. The true lengths of the various edges being set off, the remaining boundary lines of the development can then be drawn.

A regular pyramid will develop into a series of equal isosceles triangles with a common vertex. A truncated pyramid can be developed by first developing the complete pyramid as above, and then taking away the development of the portion which is cut off by the cutting plane. This

will consist in finding the true lengths of the edges from the vertex to the cutting plane, and setting off these true lengths in the development from the common vertex along the developed edges.

Problem 26.—*To develop the convex surface of a cylinder.*

The cylinder, being considered as a prism with an infinite number of sides, may be developed as the prism. The distance between the first and last element will be equal to the perimeter of the cylinder. A right section of the cylinder will develop into a right line perpendicular to the elements. If the cylinder has oblique ends the length of any element may be taken from any assumed right section and laid off in its developed position from the developed right section. Enough elements must be assumed so that the arc and chord between them are practically equal. A smooth curve drawn through the points located in the development will determine the outline of the development.

Problem 27.—*To develop the convex surface of a cone.*

If the cone is oblique it may be developed as a pyramid, by assuming points in the base such that arc and chord drawn between them are practically the same, and considering the elements drawn to these points as edges of a pyramid.

A right cone, or cone of revolution, will develop as the sector of a circle having a radius equal to the slant height of the cone, and the arc equal in length to the circumference of the case. The developed angle can be calculated from the relation

$$\frac{\text{Radius of the base}}{\text{Slant Height}} = \frac{\text{Developed Angle}}{360^\circ}$$

For oblique sections of cones of revolutions, elements are assumed, developed position found and true lengths laid off from the vertex. The curve of the developed base is drawn through the points thus found.

Exercises

148. Show the developed surface of a regular tetrahedron of 2 in. edge.

149. A pentagonal prism of 1 in. face is terminated at one end by a plane perpendicular to the axis, at the other end by a plane making 60° with the axis, so that the two shortest edges are 1¼ in. long. Show the development of the lateral surface of the prism.

150. A pyramid having a hexagonal base of 1 in. side has an edge perpendicular to the base and 2 in. long. Develop the lateral surface.

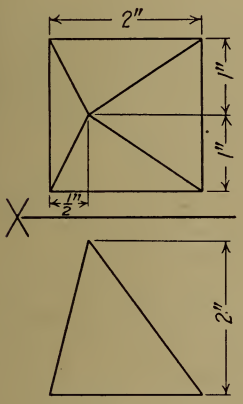


Fig. 63.

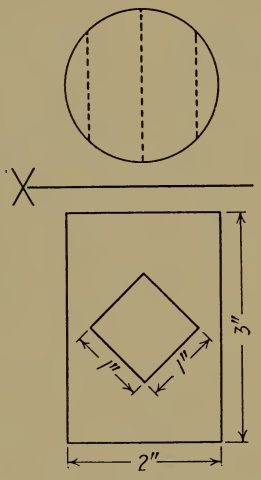


Fig. 64.

151. Develop the complete surface of the pyramid shown in Fig. 63.

152. A vertical cylinder of 1 in. diameter is intercepted between two planes, one inclined at 45°, the other at 60° to H. The two planes meet H in the same line 2 in. from the axis of the cylinder. Show the development of the convex surface of the cylinder.

153. Develop the convex surface of the cylinder which is pierced by a 1 in. square hole (Fig. 64).

154. A cone of revolution, vertex angle of 60°, is cut by a plane making 60° with the axis of the cone and cutting the axis at a point 2 in. from the vertex. Develop the convex surface.

155. A cone of revolution, vertex angle of 60°, is cut by two planes, one inclined at 60° to the axis of the cone at a point 2 in. from the

vertex, the other at 45° to the axis at a point 1 in. from the vertex. Develop the convex surface included between the planes.

156. A 2 in. circle in H is the base of a cone of 2 in. altitude, the plan of the vertex falling ½ in. outside the base. Develop the convex surface.

157. An oblique cone has for its base a 2 in. circle in V. One element of the cone is perpendicular to V and 3 in. long. Develop the convex surface of the cone.

158. A regular pyramid has a hexagon of 1 in. side as a base, and an altitude of 3 in. It is cut by a plane making 45° with the axis of the pyramid and cutting the axis at its middle point. Develop the lateral surface of the truncated pyramid.

FIG. 65.

159. In the pyramid of Exercise 158 show the projections of the shortest line that can be drawn on the surface between two points on opposite edges, one point being at a true distance of 3 in., the other 2 in., from the vertex.

160. A hexagonal nut of 1 in. side is chamfered at an angle of 60° to the axis, to a circle 1½ in. in diameter. Develop its surface.

161. A cylinder of 1½ in. diameter is terminated at one end by three planes forming a pyramid of 1 in. edge, as shown in Fig. 65. Develop the surface.

162. A cord is drawn tight over a smooth cone between two points on opposite elements, one of the points being in the base of the cone, the other halfway between the base and vertex. The altitude of the cone is 2 in., and its vertex angle 60°. Find the true length of the cord between the two points, and show projections of the cone and cord.

XI. Line Contained in Plane.

If a line lies in a plane, the traces of the line lie in the corresponding traces of the plane.

Problem 28.—*Given one projection of a line contained in a plane, to find the other projection.*

Given the plane *1* and A^h (Fig. 66). This determines the H trace of A on 1^h, and the H projection of the V trace on X. Since the V projection of the H trace lies in X and

the V trace lies in 1^v, A^v which contains these points is readily determined.

Problem 29.—*Given one projection of a point contained in a plane, to find the other projection.*

Through the given projection draw the corresponding projection of any line of the plane, that is any line having its trace in the traces of the given plane. Find the other projection of this line by the method of Problem 28, and on

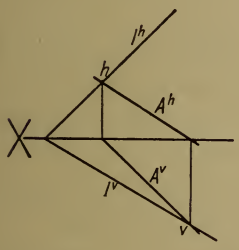


Fig. 66.

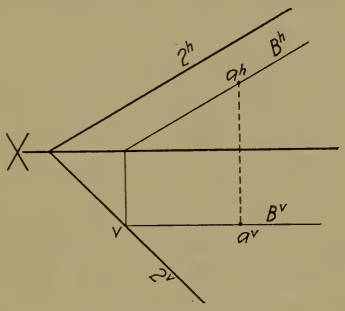


Fig. 67.

this locate the required projection of the point by a perpendicular to X through the given projection.

A convenient method is to take the auxiliary line parallel to H or to V. If parallel to H, as in Fig. 67, the V projection of the auxiliary line *B* is parallel to X and the H projection parallel to the H trace of the plane.

Problem 30.—*To find the traces of the plane containing two intersecting lines or two parallel lines.*

Note.—If two lines in space intersect, the point of intersection of the H projections and the point of intersection of the V projections, being the two projections of the same point, must lie in the same perpendicular to X.

Since the traces of a line must lie in the corresponding traces of the containing plane, the line joining the H traces of the two lines will be the H trace of the plane. Similarly the line joining the V traces of the two lines will be the V

trace of the plane. The two traces of the plane must of course meet in X.

Problem 31.—*To find the traces of the plane containing a given point and a given line.*

Draw a line joining the given point with any point of the given line. The plane of these two lines is the one required, and its traces can be found by the method of the preceding problem.

Problem 32.—*To find the traces of the plane containing three given points.*

Connect the points by lines so as to form two intersecting lines. The plane of these lines is the one required.

Problem 33.—*To find the traces of the plane containing a given point and parallel to two given lines.*

Through the given point draw lines parallel to the given lines. The plane of these lines is the one required.

Problem 34.—*To find the traces of the plane containing a given line and parallel to a second given line.*

Through any point of the first given line draw a line parallel to the second given line. The plane of these two lines is the one required.

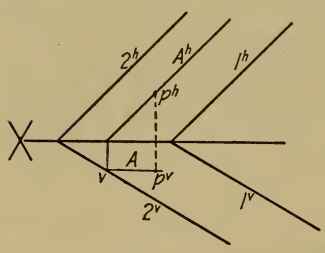


Fig. 68.

Problem 35.—*To find the traces of the plane containing a given point and parallel to a given plane.*

Through the given point draw a line parallel to any line of the given plane. The plane containing this line and having its traces parallel to those of the given plane is the one required. A convenient auxiliary line to use is one parallel to one of the traces of the given plane.

In Fig. 68 p is the given point and 1 the given plane. Line A is drawn through p and parallel to 1^h. A^v will therefore be parallel to X. The V trace of line A is a point in

the V trace of the required plane. Its traces are drawn parallel to 1^v and 1^h.

Problem 36.—*To find the lines of a given plane which make a given angle with H or V.*

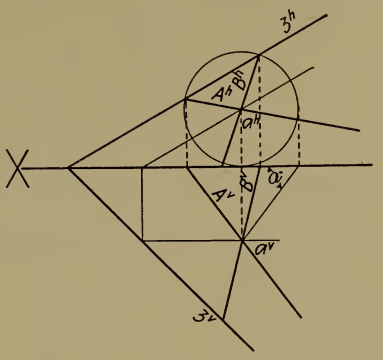


Fig. 69.

Note.—The angle which a line of a plane makes with H or V cannot be greater than the angle which the plane itself makes with H or V.

Let it be required to find the lines of plane *3* which make an angle α with H (Fig. 69). Take any point of the given plane as the vertex of a cone with its base in H and its elements making the given angle with the plane of the base. The points of intersection of the circle of the base with the H trace of the plane will indicate the elements of the cone which coincide with the plane, and hence *A* and *B* are the desired lines.

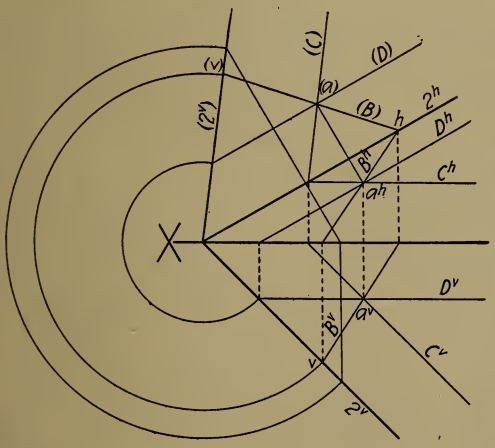


Fig. 70.

Problem 37.—*To find the projections of a point or line of a given plane when the point or line has been located in the revolved position of the plane.*

Suppose that the plane *2* has been revolved into H and a

point (a) located in the revolved position. Required to counter revolve the plane and find the projections of a.

First Method (Fig. 70).—Through (a) draw any line of the plane, such as (B) and locate its revolved V trace (v). Counter revolve this to v, and both projections of line B may be drawn, as we now have both its traces. Through (a) draw a perpendicular to the axis of revolution 2^h, which determines a^h on B^h. a^v is located on B^v by a perpendicular from a^h. Lines C (parallel to 2^v) and D (parallel to 2^h) show two other lines either one of which may conveniently be used instead of B.

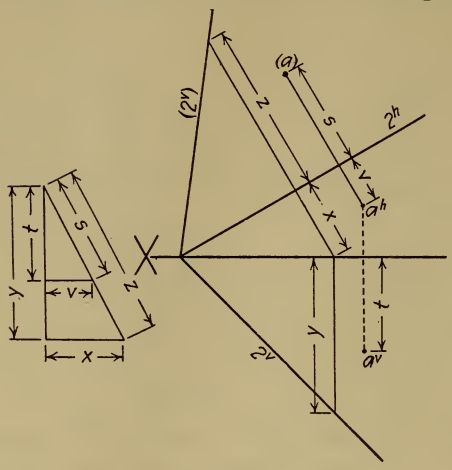


Fig. 71.

Second Method (Fig. 71).—This method makes use of the right triangle construction of Problem 7, page 19. If a number of different points lying in a plane are revolved into H about the H trace of the plane as axis, then all the right triangles used in the construction are similar. In Fig. 71 suppose that plane 2 has been revolved into H by the right triangle method. Then take (a) as the revolved position of the point whose projections are required. From (a) draw a perpendicular to 2^h. Take the distance s and lay it off along the hypotenuse of the right triangle, and complete a right triangle similar to the original one.

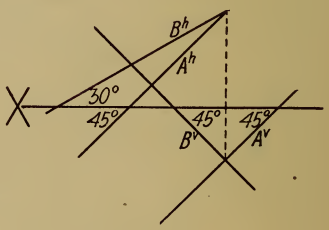


Fig. 72.

The distances t and v will then locate a^h and a^v as shown in the figure.

In general the first method is more convenient for locating a single point, while the second method may be found better when a considerable number of points are to be located.

Exercises

163. Show the traces of the plane $\overline{AB}$ (Fig. 72).

164. The line B is parallel to X, ½ in. from H and 1 in. from V. The point a is ½ in. from V and 1½ in. from H. Find the traces of the plane $\overline{aB}$.

165. Does point a lie in plane *2* (Fig. 73)?

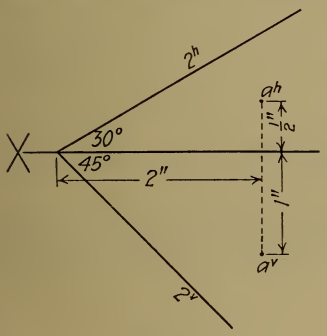


Fig. 73.

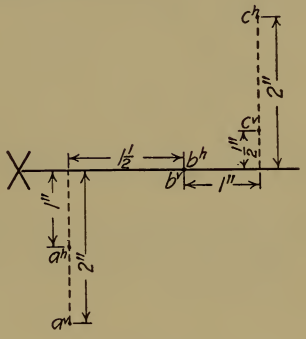


Fig. 74.

166. Show the traces of the plane $\overline{abc}$ (Fig. 74).

167. Triangle abc (ab = 1 in., ac = 2½ in. and bc = 2 in.) lies in a plane inclined at 60° to V. ab is parallel to V and ½ in. from it. ac is parallel to H. Show the projections of the triangle.

168. Find the traces of the plane containing the point e and parallel to the lines C and D (Fig. 75).

169. Find the traces of the plane containing the line C and parallel to the line D (Fig. 75).

170. Plane *3* has its H trace inclined at 45° to X and its V trace at 30°. Find a point in the plane which is 1½ in. from V and 1 in. from H.

171. A rectangle 1 in. × 3 in., with the long side inclined at 30° to X, is the plan of a square the upper edge of which is ¼ in. below H. Show the elevation of the square and the traces of the plane containing it.

172. Plane *4* contains point *c* (Fig. 76). Find 4^v.

173. Find a line of plane *2* of Fig. 73, the line to be inclined at equal angles to H and V.

174. Find the traces of the plane $\overline{cD}$ (Fig. 77).

175. Find the revolved positions of *c* and *D* when the plane $\overline{cD}$ is revolved into H about its H trace (Fig. 77).

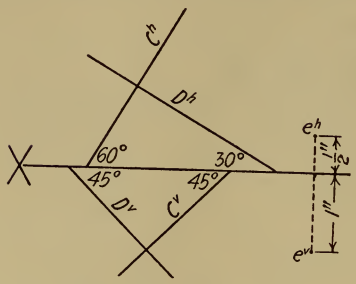


FIG. 75.

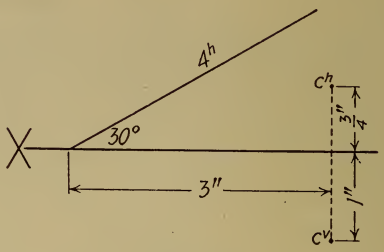


FIG. 76.

176. Point *p*, lying in plane *2* and ½ in. from V, moves to the position (*p*) when the plane is revolved into H (Fig. 78). Find 2^v and p^v.

177. Plane *2* has its H trace inclined at 30° to X and its V trace at 45°. Show the projections of a line lying in the plane, parallel to H and 1 in. from it.

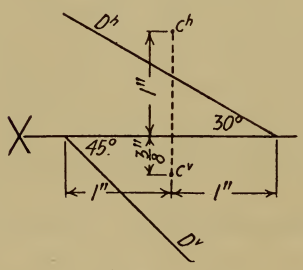


FIG. 77.

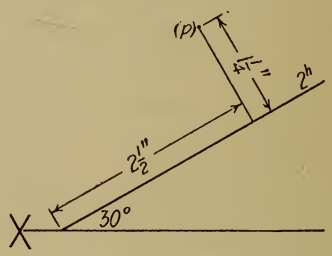


FIG. 78.

178. A point in plane *2* of Exercise 177, is ½ in. from 2^h and 1 in. from 2^v. Show the projections of the point.

179. A triangle in H, sides 1½ in., 1 in. and 2 in., with the 2 in. side parallel to X, is the plan of a triangle situated in a plane which is inclined at 60° with H and 45° with V. Show the elevation of the triangle.

180. Find the angle K of the plane $\overline{abc}$ (Fig. 79).

181. 5^h and 5^v are parallel to X and each 2 in. from it. Through a point of the plane $\frac{3}{4}$ in. from H draw a line of the plane, the line to make 30° with H.

182. Find the traces of the plane $\overline{AB}$ (Fig. 80).

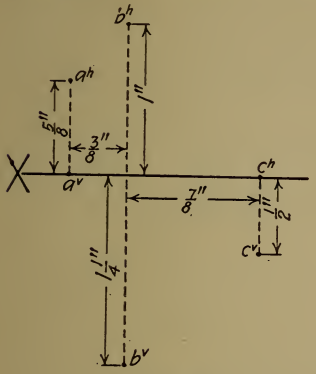


Fig. 79.

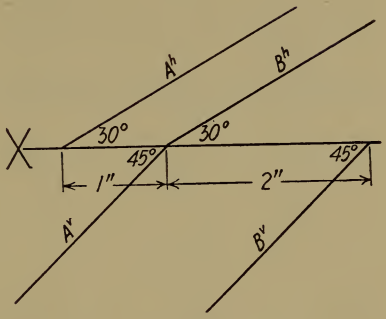


Fig. 80.

183. Find the traces of the four planes of the tetrahedron of Fig. 49.

184. Line A lies in plane 3 (Fig. 81). Show the projections of a second line of the plane, intersecting A at a true angle of 45°.

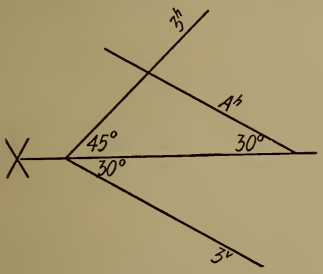


Fig. 81.

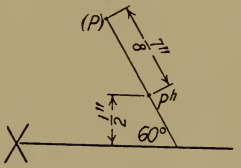


Fig. 82.

185. Show the projections of a regular hexagon of 1 in. side lying in plane 3 of Fig. 75, with one side in 3^h.

186. Point p, whose H projection is shown in Fig. 82, lies in a plane whose angle $K = 45°$. When the plane is revolved into H the point moves to the position (p). Find the traces of the plane and the V projection of the point.

187. Point p, whose H projection is shown in Fig. 82, lies in a plane

whose angle $\theta = 45$. When the plane is revolved into H the point moves to the position (p). Find the traces of the plane and the V projection of the point.

188. A regular tetrahedron has edges $2\frac{1}{2}$ in. long. Find the lines of a face which make 60° with the plane of the base.

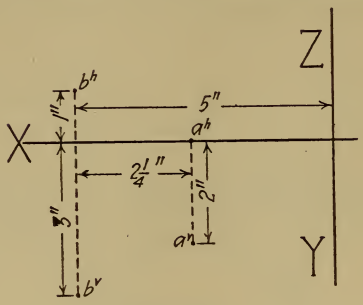


Fig. 83.

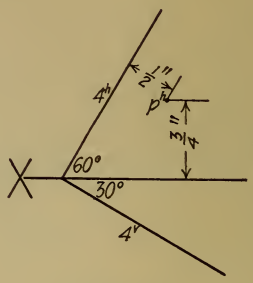


Fig. 84.

189. A ray of light from b is reflected from V at the point a, to P and thence to H. Find the points of meeting P and H, and the angle at which the ray meets H (Fig. 83).

190. Find the lines of plane 4 (Fig. 84), passing through p and making an angle of 45° with V.

191. A point p is $2\frac{1}{2}$ in. from H and V. Draw through p two

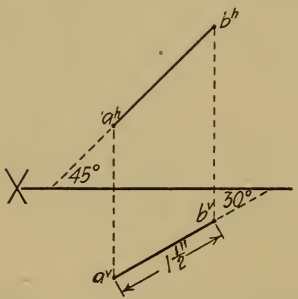


Fig. 85.

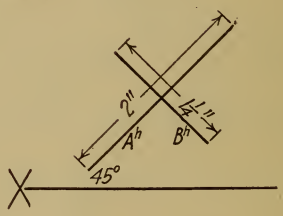


Fig. 86.

lines each making 60° with H, the plans of the lines making 120° with each other. Find the angle between the lines.

192. Plane 2 has its H trace at 30° to X and its V trace at 45°. A point $\frac{1}{2}$ in. from X and $2\frac{1}{2}$ in. from 2^v is the elevation of a point

which is ½ in. from V. Show the traces of the plane parallel to 2 and containing the point.

193. Through a pass a plane whose traces are perpendicular to the corresponding traces of plane 2 (Fig. 73).

194. Find a line of plane 2 of Exercise 192, which will make 45° with X when 2 is revolved into H.

195. Show the projections of a line of plane 2 of Exercise 192, which will make the same angle with X when the plane is revolved into H about its H trace as it does when the plane is revolved into V about its V trace.

196. Construct a square on the diagonal ab, having its plane vertical (Fig. 85).

197. A and B (Fig. 86), are the principal axes of the H projection of

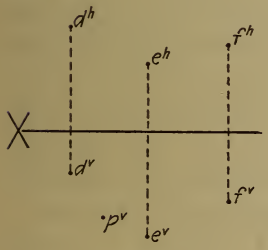


Fig. 87.

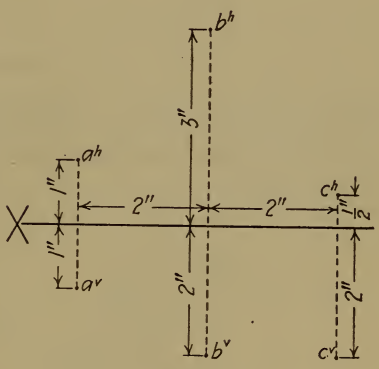


Fig. 88.

a circle. Show the principal axes of the V projection of the same circle.

198. The point p lies in the plane $\overline{def}$ (Fig. 87). Find its H projection without using the traces of the containing plane.

199. Show the projections of a line parallel to V and ¾ in. from it, lying in the plane $\overline{abc}$ (Fig. 88). Do not use the traces of the containing plane.

200. Find the projections of a line passing through b, inclined at 60° to V and lying in the plane $\overline{abc}$ (Fig. 88). Do not use the traces of the containing plane.

201. Show the projections of the bisector of the angle acb of the triangle of Exercise 82.

XII. Intersections.

Problem 38.—*To find the line of intersection of two given planes.*

The required line lies in both given planes, therefore its H trace must lie in the H traces of both planes, or at their

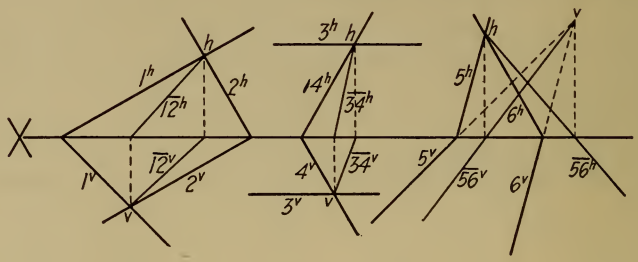


Fig. 89.

point of intersection. Similarly, the V trace of the line of intersection is at the point of intersection of the V traces of the planes.

Case a.—When the corresponding traces intersect within the limits of the drawing (Fig. 89). The line is readily determined from its traces.

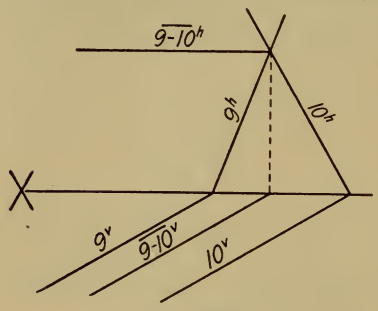


Fig. 90.

Case b.—When one pair of traces is parallel, the other intersecting (Fig. 90). The pair of intersecting traces gives one point of the required line. The direction of this line will be parallel to the parallel traces; for if two intersecting planes intersect a third plane in parallel lines, all three intersections will be parallel. Therefore $\overline{9\text{–}10^v}$ is drawn parallel to 9^v and 10^v, and $\overline{9\text{–}10^h}$ parallel to X, that is parallel to the H projections of 9^v and 10^v.

Case c.—When one plane is parallel to H or V. In

Fig. 91 plane 7 is taken parallel to V. The pair of intersecting H traces gives the H trace of the required line. The direction of this line will be parallel to the V trace of the other plane; for if two parallel planes are intersected by a third plane, their lines of intersection are parallel.

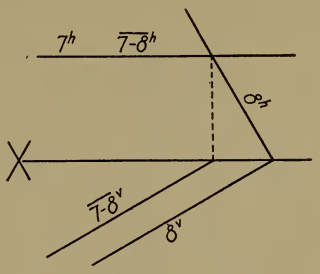


Fig. 91.

Case d.—When one pair of traces does not intersect within the limits of the drawing, or when all four traces meet X in the same point (Fig. 92). As in the previous case, one point of the line is determined by the intersecting traces. A second point can be found by assuming an auxiliary plane, preferably perpendicular or parallel to one of the planes of projection, intersecting both the given planes and cutting a line from each. These lines meet in a point com-

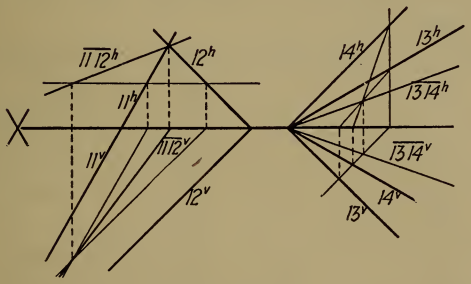


Fig. 92.

mon to all three planes, and hence a point of the required line of intersection.

Case e.—When neither pair of traces meets within the limits of the drawing (Fig. 93). In this case two auxiliary planes are taken, giving two points in the required line of intersection.

Problem 39.—*To find the point of intersection of a given line and a given plane.*

If any auxiliary plane is passed through the line, this plane will intersect the given plane in a line which contains the required point. It is generally most convenient

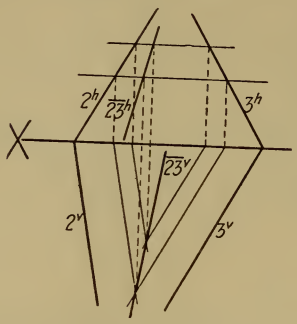


Fig. 93.

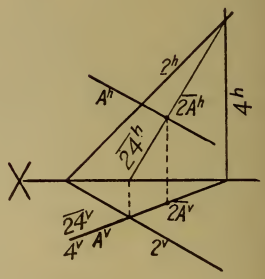


Fig. 94.

to take the auxiliary plane perpendicular to either H or V, in which case one trace of the plane coincides with the corresponding projection of the line, and the other trace is perpendicular to X.

In Fig. 94, *2* is the given plane, *A* the given line and the auxiliary plane *4* is taken perpendicular to V. The line of intersection of *2* and *4* meets the given line *A* in the required point $\overline{2A}$. When the auxiliary plane is taken perpendicular to H or V, then one trace of the plane and the corresponding projections of the given line and of the line of intersection coincide. In the figure, A^v, 4^v and $\overline{24}^v$ coincide, so that the required point must be located from the H projections, and its V projection determined by a perpendicular to X.

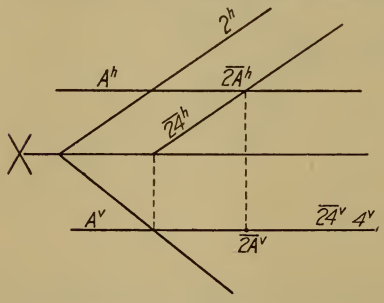


Fig. 95.

Special Cases.—If the given line is in a position nearly parallel to H and V, it may happen that the auxiliary plane will not have its traces intersecting the corresponding traces of the given plane within the limits of the drawing. In this case the line of intersection must be found by the methods of Problem 38, case e.

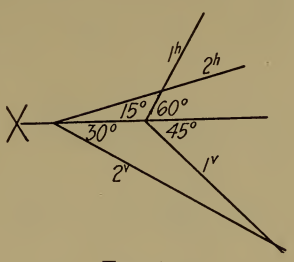


Fig. 96.

If the given line is parallel to H and V, the auxiliary plane will be parallel to one plane of projection. The line of intersection with the given plane will then be found by the method of Problem 37, case c. This is illustrated in Fig. 95, in which *A* is the given line, *2* the given plane and *4* the auxiliary plane parallel to H. Notice the similarity with Fig. 67.

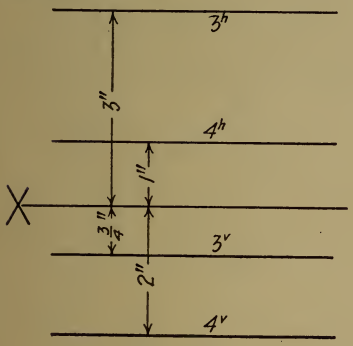


Fig. 97.

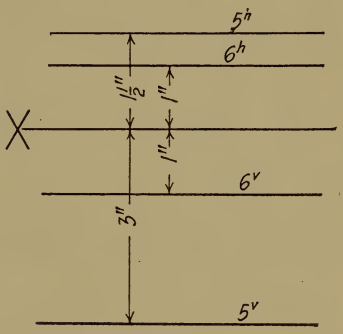


Fig. 98.

Exercises

202. Show the line of intersection of the planes *1* and *2* (Fig. 96).

203. Show the line of intersection of the planes *3* and *4* (Fig. 97).

204. Show the line of intersection of the planes *5* and *6* (Fig. 98).

205. Show the line of intersection of the planes *7* and *8* (Fig. 99).

206. Show the point of intersection of the plans *2*, *3* and *4* (Figs. 96 and 97).

207. Show the line of intersection of planes *9* and *10* (Fig. 100).

208. Find the point of intersection of planes *2*, *3* and *4* (Figs. 108, 101 and 104).

209. Find the point of intersection of line *A* with plane *3* (Fig. 101).

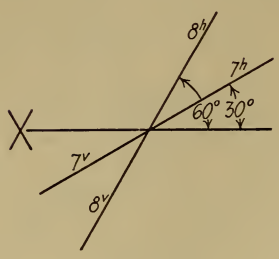


Fig. 99.

210. Line *A* is parallel to X, 1 in. from V and 2 in. from H. Find its point of intersection with plane *2* (Fig. 96).

211. 4^h is inclined at 45°, and 5^h at 60° to X. θ for 4 is 30°, and for *5* is 45°. Find ϕ for the line $\overline{45}$.

212. Find the point of intersection of line *A* with plane *2* (Fig. 102).

213. Find the point of intersection of line *B* with plane *3* (Fig. 103).

214. Find the point of intersection of line *C* with plane *4* (Fig. 104).

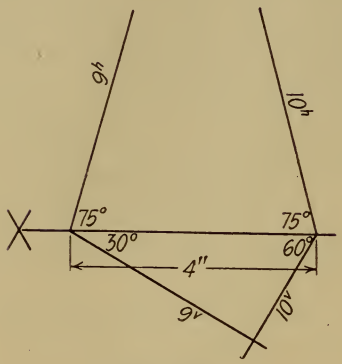


Fig. 100.

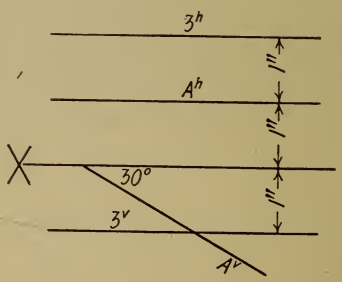


Fig. 101.

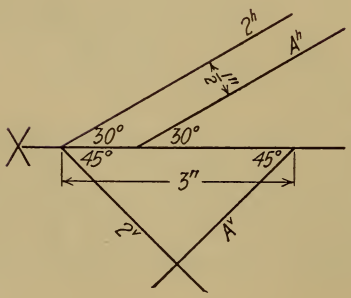


Fig. 102.

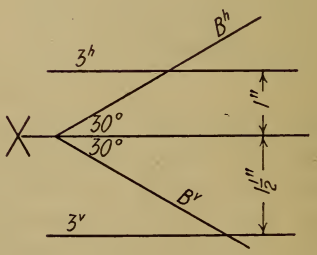


Fig. 103.

215. What length of line parallel to X, 1 in. from H and 2 in. from V is intercepted between planes 7 and 8 (Fig. 105)?

216. Show the projections of the point $\overline{9A}$ (Fig. 106).

217. The plane 7 of Fig. 105 is intersected by a line whose projections cross at a point ½ in. below X and 1 in. from 7^v, and are

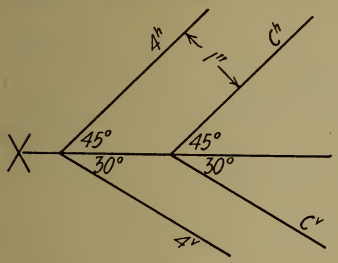


Fig. 104.

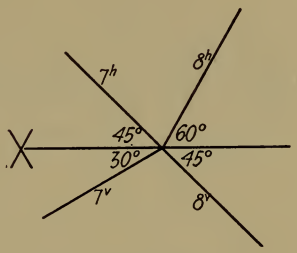


Fig. 105.

parallel to the corresponding traces of the plane. Find the point of intersection.

218. The H trace of a plane which contains the point *a*, 1 in. from H and 2 in. from V, makes 75°, and the V trace 30° with X. A second plane also contains the point *a* and has its traces perpendicular to the

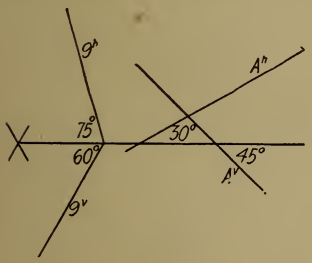


Fig. 106.

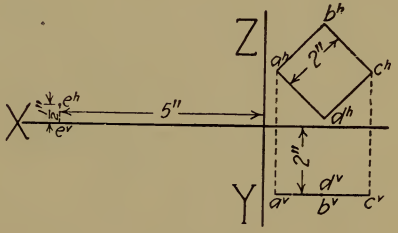


Fig. 107.

corresponding traces of the first plane. Find the line of intersection of the two planes.

219. With the point of sight at *e*, show the appearance of the square *abcd* on the plane P (Fig. 107).

220. The base *abc* of a regular triangular pyramid is in V and of 2 in. side, the side *ab* being inclined at 15° to X. Show projections of the vertex *d* when the face *abd* is inclined at 30° to H.

221. A shaft *A* is 5 ft. above and a shaft *B* is 5 ft. below a floor.

The distance between the centers of the shafts is 16 ft., and they are inclined at 30° to a vertical wall. An 8 ft. pulley on *B*, clearing the wall by 6 in., drives a 4 ft. pulley on *A*, on the other side of the

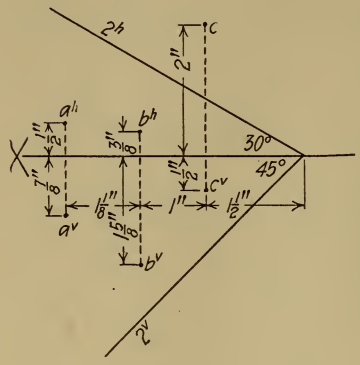


Fig. 108.

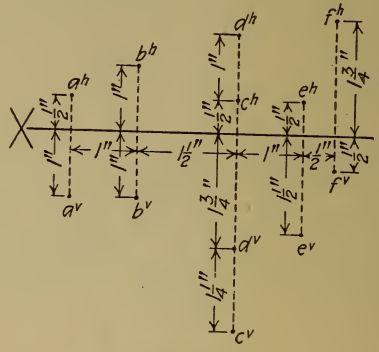


Fig. 109.

wall, by a belt 2 ft. wide. Show where the wall and floor are to be cut for the belt. Scale, ¼ in. = 1 ft. 0 in.

222. Find the line of intersection of plane *2* with the plane $\overline{abc}$ (Fig. 108). Do not use the traces of the containing plane.

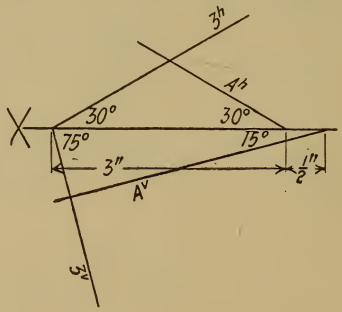


Fig. 110.

223. Find the line of intersection of the planes $\overline{abc}$ and $\overline{def}$ (Fig. 109). Do not use the traces of the containing planes.

224. Find the point of intersection of line *A* with plane *3* (Fig. 110).

XIII. Perpendiculars.

If a line is perpendicular to a plane its projections are perpendicular to the corresponding traces of the plane.

Problem 40.—*Through a given point, to draw a line perpendicular to a given plane.*

Through the given projections of the point draw the projections of the required line perpendicular to the corresponding traces of the given plane.

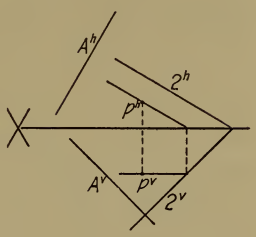


Fig. 111.

Problem 41.—*To pass a plane through a given point and perpendicular to a given line.*

Given point p and line A, Fig. 111, to find the traces of a plane containing p and perpendicular to A.

The directions of the traces of the required plane will be perpendicular to the projections of the given line. Hence this problem is similar to Problem 35. Through the given point draw a line parallel to one of the traces. In the figure line B is drawn parallel to the H trace, so that its V projection is parallel to X. The V trace of this line is a point on the V trace of the required plane, and both traces may be drawn in the proper directions.

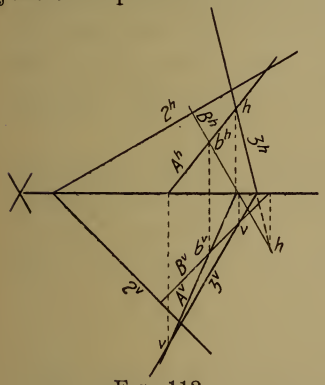


Fig. 112

Problem 42.—*To pass a plane through a given point and perpendicular to two given planes.*

First Method.—Through the given point draw lines perpendicular to the given planes. The plane of these lines is the one required.

Second Method.—Find the line of intersection of the two given planes. The required plane is perpendicular to this line of intersection, and its traces may be found by the method of Problem 41.

Problem 43.—*Through a given line to pass a plane perpendicular to a given plane.*

Given line A and plane *2*, Fig. 112, to pass a plane through A and perpendicular to *2*.

From any point of the line, such as b, draw a line B perpendicular to the plane *2*. The plane of the lines A and B is the required plane.

NOTE.—Perpendicular planes do not have their corresponding traces perpendicular. One pair of traces may be perpendicular provided that one of the planes is perpendicular to that plane of projection.

Problem 44.—*Through a given point to draw a line perpendicular to a given line.*

First Method.—Find the plane containing the point and line. Revolve this plane into H or V, and with it the point and line. In the revolved position draw the required line perpendicular to the given ine. Counter revolve to find its projections.

Second Method.—Through the given point pass a plane perpendicular to the given line. Find the point of intersection of the plane and line. The line joining the given point with this point of intersection is the required line.

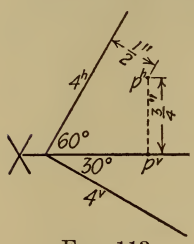


FIG. 113.

NOTE.—Perpendicular lines do not have their corresponding projections perpendicular. One pair of projections may be perpendicular provided one of the lines is parallel to that plane of projection.

Exercises

225. Show the point of intersection of the line passing through p and perpendicular to the plane *4* with the plane *4* (Fig. 113).

226. Plane *3* has its traces appearing as one straight line at 45° to X. A point t in plane *3* is 1 in. from 3^h and 1½ in. from 3^v. Show the H trace of a perpendicular to *3* through t.

227. The H traces of the planes *2* and *3* form with X an equilateral triangle. *2* is inclined at 30° and *3* at 75° with H. Show the traces of a plane perpendicular to *2* and *3*.

228. The V trace of a plane is inclined to X at 60° and the H trace at 45°. Through a point in this plane which is 2 in. from H and 1 in. from V pass a plane which shall be perpendicular to the given plane and also to V.

229. Three lines, *A*, *B* and *C*, meet at a point *o*. *B* makes an

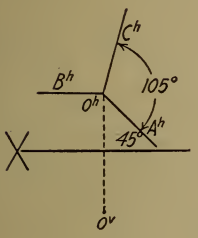


Fig. 114.

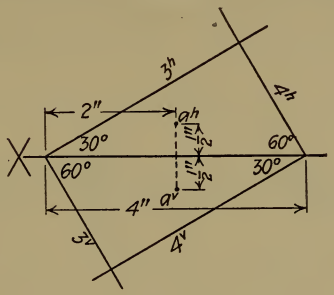


Fig. 115.

angle of 45° with H. *A* is perpendicular to the plane $\overline{CB}$. What angle does *C* make with H (Fig. 114)?

230. Find the traces of the plane which contains point *a* and is perpendicular to planes *3* and *4* (Fig. 115).

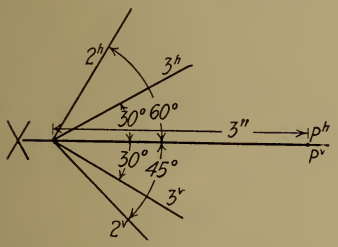


Fig. 116.

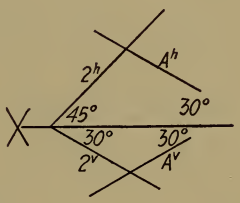


Fig. 117.

231. A 1 in. square lies in V with sides inclined at 30° and 60° to X. Project the square on a plane whose traces are inclined at 45° to X.

232. What length of line through *p* and perpendicular to plane *2* is intercepted between the planes *2* and *3* (Fig. 116)?

233. Show the traces of the plane containing point *p* and perpendicular to planes *2* and *3* (Fig. 116).

234. Show the traces of the plane containing *A* and perpendicular to *2* (Fig. 117).

235. If a line and a plane are perpendicular, their angles of inclination with H are complementary, likewise their angles of inclination with V. Check this by finding the projections of a line making 30° with H and 45° with V; draw a plane perpendicular to the line and find θ and ϕ for the plane.

236. A cone of revolution has its vertex at a, its base in plane *2* and base angle 45°. Show its projections (Fig. 118).

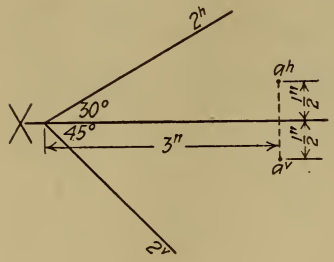


Fig. 118.

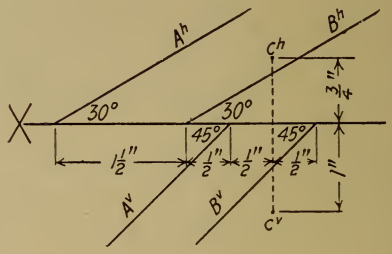


Fig. 119.

237. Show the projections of the line passing through c and perpendicular to plane $\overline{AB}$ (Fig. 119).

238. Find the traces of the plane passing through a, parallel to A and perpendicular to *2* (Figs. 118 and 119).

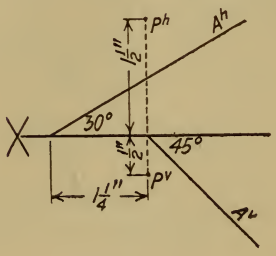


Fig. 120.

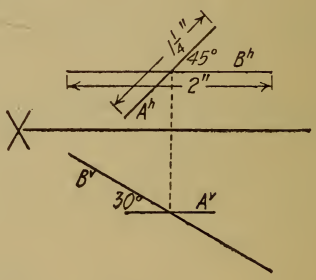


Fig. 121.

239. Show the projections of the line through p and perpendicular to line A (Fig. 120).

240. The H projection of a square having a side in H and an adjacent side in V is a rectangle having sides of 1½ in. and ¾ in. Project this square on the plane *3* (Fig. 113).

241. The H projection of the axis of a square base pyramid is inclined at 30°, and the V projection at 45° to X. A corner of the

base is in H and the adjacent edges of the base are inclined at 30° and 60° to the H trace of the plane of the base. Show the projections of the pyramid.

242. *A* and *B* are the diagonals of the base of a pyramid of 2 in. altitude. Its axis is perpendicular to the plane of the base. Show projections of the pyramid (Fig. 121).

XIV. Angles.

Problem 45.—*To find the angle between two intersecting lines.*

First Method.—Find the plane of the two lines and revolve it with the lines into or parallel to H or V, whence the angle will appear in its true size.

Second Method.—Draw a line intersecting the two lines and forming a triangle. Find the true length of each side and construct the triangle, which gives the required angle. In Fig. 122 let it be required to find the angle between the intersecting lines *A* and *B*. Draw any third line *C* intersecting *A* and *B*. Finding the true length of *C* is simplified by drawing *C* parallel to H or V. In the figure it is drawn parallel to V, therefore C^v shows true length. Find the true lengths of *A* and *B*, construct the triangle with the three true lengths as sides, and the angle between *A* and *B* is the required angle α.

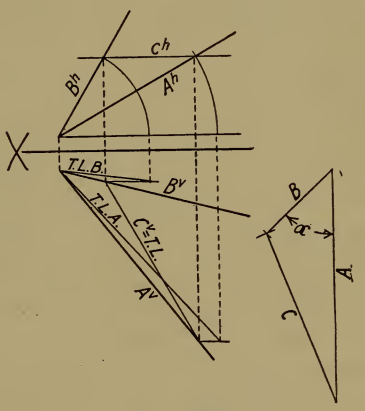


Fig. 122.

Problem 46.—*To find the angle between two given planes.*

First Method.—Pass a plane perpendicular to the line of intersection of the given planes. Find the lines cut from the given planes by this plane, and the angle between

these lines, which may be found by the methods of Problem 45, is the angle required.

Second Method.—From any point in space draw two lines, one perpendicular to each plane. The angle between these lines, which may be found by the methods of Problem 45, is the supplement of the angle required.

Problem 47.—*To find the angle between a given line and a given plane.*

From any point of the given line draw a perpendicular to the plane. The angle between the given line and the perpendicular, which may be found by the methods of Problem 44, is the complement of the angle required.

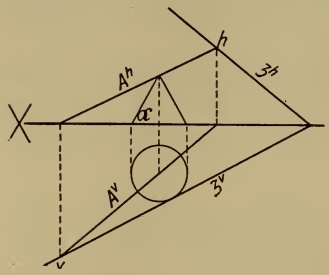


Fig. 123.

Problem 48.—*To pass a plane through a given line and making a given angle with H or V.*

Note.—The angle which the plane makes with H or V cannot be less than that which the line makes with H or V.

Given line A, Fig. 123, to pass a plane through A making a given angle α with V. With any point of A as vertex construct a cone with base in V and base angle α. Draw 3^v through the V trace of A and tangent to the base of the cone, and 3^h through the H trace of A. This gives the required plane. In general two solutions are possible, according as the V trace is drawn tangent above or below the base of the cone.

Exercises

243. What is the angle between the diagonal of a cube and an edge?

244. θ for 2 is 45°, for 3 it is 60°, for the line $\overline{23}$ it is 30°. Find the angle between the planes.

245. Two intersecting lines A and B determine a plane. θ for A

is 45°, for B 30° and for the plane 60°. Find the angle between the lines.

246. The planes *2* and *3* are at right angles to each other. *2* is inclined at 60° to H. The line $\overline{23}$ is inclined at 45° to H. What angle does *3* make with H?

247. Find the angle between the planes *5* and *6* (Fig. 124).

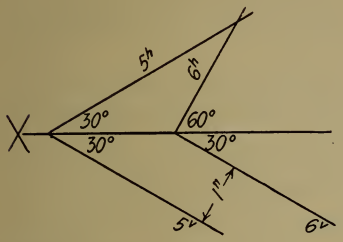


Fig. 124.

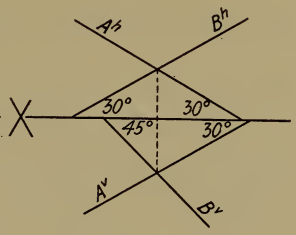


Fig. 125.

248. Find the angles between A and $\overline{AB^h}$ and between A and AB^v (Fig. 125).

249. Find the traces of the plane containing A and inclined at 60° to the plane $\overline{AB}$ (Fig. 125).

250. A rectangular block measures 1½ in. × 2 in. × 3 in. What angle does a line drawn from a corner to the center of the block make with the plane determined by the three adjacent corners?

251. A pyramid having a triangular base of 2 in. side has dihedral angles between its faces of 75°. What is its altitude?

252. A plane is inclined at 60° to H and V. At what angle is it inclined to X?

253. Find the true angle between plane *2* and line A (Fig. 126).

254. Line B is inclined at 30° to H, parallel to V and 1 in. from it. Find the traces of a plane containing the line and inclined at 45° to V.

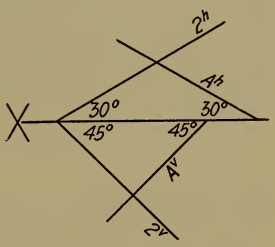


Fig. 126.

255. Two lines intersect at a point 1½ in. behind V and 1 in. below H. Both lines are inclined at 30° to H, one at 45° and the other at 30° to V. Find the angle between the lines.

256. The H trace of a plane makes 60° with X, and the plane makes 60° with H. This plane contains two lines, one parallel to H and the other inclined at 45° to H. Find the angle between the lines.

257. Plane *4* has both traces inclined at 45° to X. Line *C* is perpendicular to H. Find the angle between them.

258. A ray of light from *a* is reflected from *c* to *b* (Fig. 127). Show the traces of the reflecting plane.

259. Given the point and plane of Exercise 228, draw through the point a line making 30° with the plane and parallel to H.

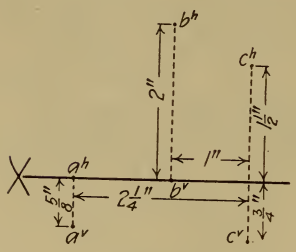


Fig. 127.

260. Through line *A* pass a plane such that *A* makes 60° with the H trace of the plane (Fig. 119).

261. Show the projections of a line containing point *p* and intersecting line *A* at an angle of 45° (Fig. 120).

262. Find the dihedral angles between the faces of the pyramid of Fig. 63.

263. Show the traces of the plane which bisects the dihedral angle between planes *5* and *6* (Fig. 124).

264. Show the projections of a line passing through *a*, making 60° with H and perpendicular to plane *2* (Fig. 118).

265. A line passing through the corner of a cube makes 75° with one edge and 60° with another. What angle does it make with the third edge?

266. 2^h is inclined at 45° and 2^v at 30° to X. The line of intersection of the planes *2* and *3* bisects the angle *K* for the plane *2*. The plane *3* is inclined at 30° to H. Show its traces.

267. The traces of a plane make 45° with X. Find the traces of a second plane which is inclined at 60° to H and is perpendicular to the first. Find also the line of intersection of the two planes.

268. Find the traces of a plane containing line *A* and inclined at 60° to H (Fig. 119).

XV. Distances.

Problem 49.—*To find the distance from a given point to a given plane.*

Through the given point draw a perpendicular to the given plane and find the point of intersection of the perpendicular with the plane. Find the distance between this point and the given point as in Problem 10.

Problem 50.—*To find the distance from a given point to a given line.*

Draw a line from the given point perpendicular to the given line, as in Problem 44. The true length of this line is the distance required.

Problem 51.—*To find the distance between two parallel planes.*

First Method.—Construct a third plane perpendicular to the two given planes, and find their lines of intersection. Revolve the third plane with the lines of intersection into H or V, and the distance between the lines is the required distance. For convenience take the third plane perpendicular either to H or V.

Second Method.—Draw a line perpendicular to the given planes, and find its point of intersection with each. The distance between these points is the required distance.

Problem 52.—*To find the distance between two parallel lines.*

First Method.—Find the plane of the two lines and revolve the plane, and with it the lines, into H or V. The distance between the revolved lines is the required distance.

Second Method.—Pass a plane perpendicular to the given lines, and find its point of intersection with each. The true distance between these points is the required distance.

Problem 53.—*To find the shortest distance between two lines not in the same plane.*

Pass a plane through one of the lines and parallel to the other. From any point of the other line draw a perpendicular to the plane. The true length of this perpendicular is the required distance.

Exercises

269. What is the distance from a corner of a 2 in. cube to the plane of the three adjacent corners?

270. Find the distance from a to B (Fig. 128).

271. Find the shortest distance between the lines A and B (Fig. 129).

272. 2^h and 3^h are parallel, 1½ in. apart and inclined at 45° to X. The V traces are also parallel and 1 in. apart. What is the distance between the planes?

273. Find the distance from point a to plane 2 (Fig. 130).

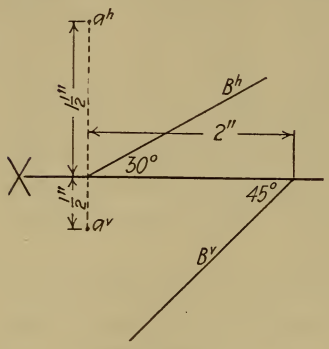


Fig. 128.

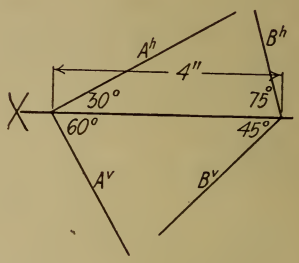


Fig. 129.

274. Find the distance between planes 3 and 4 (Fig. 131).

275. The H projections of two points are 1 in. apart, and the V projections are 2 in. apart. What is the greatest and what the least distance apart that the points can be placed?

276. Find the distance between lines A and B (Fig. 119).

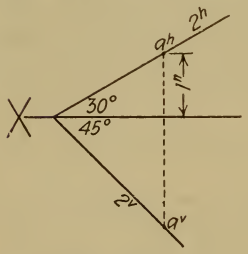


Fig. 130.

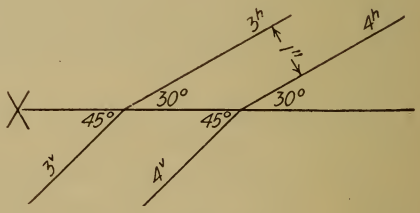


Fig. 131.

277. A line 3 in. long between its H and V traces has its H and P traces at the same point. The line makes an angle of 45° with H and 15° with V. Find a point of the P plane which is 2½ in. from both ends of the line.

278. A triangular pyramid has a base of 3 in. side and an altitude

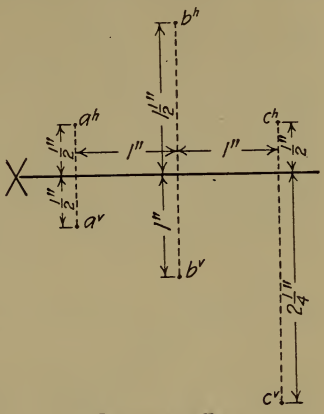
of 4 in. Show the projections of a point which shall be respectively ½ in., ¾ in. and 1 in. from the three faces of the pyramid.

279. Find the projections of a line parallel to X which is equally distant from the points *a*, *b* and *c* (Fig. 132).

280. The line *A* is inclined at 30° to H and 45° to V. The line *B* is inclined at 45° to H and 30° to V. The shortest distance between the two lines is 2 in. Show their projections.

281. Point *a* is 1½ in. from V and ½ in. from H. Point *b* is 2 in. to the right, ½ in. from V and 1½ in. from H. Plane *3* contains point *a* and has its H trace at 45° and its V trace at 30° to X. Find the distance from the point *b* to the plane.

282. Find the distance from point *c* to plane $\overline{AB}$ (Fig. 119).



[Fig. 132.]

283. Point *p* is at a true distance of 1 in. from plane *2* (Fig. 133). Find 2^v.

284. Show the traces of a plane parallel to plane *2* and at a distance of 1 in. from it (Fig. 130).

285. Line *A* is parallel to V and 1½ in. from it, and inclined at 30° to H. Point *b* is 1¼ in. from V and ½ in. from H, b^v being 1 in. from A^v. Show projections of the path of *b* when it is revolved about *A* as axis.

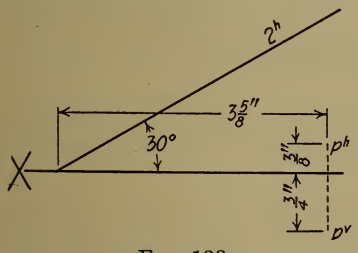


Fig. 133.

286. Find the distance from point *d* to the plane of *a*, *b* and *c*, Fig. 109, without using the traces of the plane $\overline{abc}$.

287. A cube of 2½ in. edge has its center 2 in. from H. An edge of the cube has one end ½ in. and the other 1½ in. from H. Show projections of the cube.

XVI. Tangent Planes.

If a straight line is tangent to a curve, two consecutive points on the curve lie in the straight line, the two points being infinitely close together. Similarly if a line is tangent

to a surface, it has two consecutive points in common with the surface. These two points, being infinitely close together, constitute what is commonly referred to as the point of tangency.

If two lines are tangent to a surface at the same point, the plane of the lines is tangent to the surface, and the plane has three points in common with the surface. These points, being infinitely close together, constitute what is commonly referred to as the point of tangency.

If the surface is a ruled surface, the tangent plane will contain two consecutive elements of the surface, and these elements being infinitely close together constitute the line of tangency. In the case of doubly ruled surfaces of which there are but two, the hyperboloid and the hyperbolic paraboloid, the tangent plane at a given point of the surface is determined by the two elements through the point.

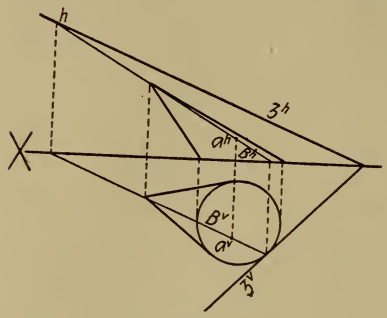


Fig. 134.

Problem 54.—*Given one projection of a point on the surface of a cone, to pass a plane tangent to the cone and containing the point.*

Consider the general case of an oblique cone, and place its base in V, as shown in Fig. 134. Let a^h be the given projection of a point on the surface of the cone. Draw the projections of the element containing this point, as B^h and B^v. The V trace of the required plane, 3^v is tangent to the base of the cone at the V trace of B, and 3^h passes through the H trace of B. If the H trace of B should fall without the limits of the drawing or be inconvenient to find, any auxiliary line drawn through the vertex of the

cone and a point of the V trace of the plane may be drawn, and its H trace used instead.

Problem 55.—*To pass a plane tangent to a given cone and containing a given point outside its surface.*

Suppose the base of the cone assumed in H. The tangent plane will contain the line drawn through the vertex of the cone and the given point. Hence the H trace of the tangent plane will pass through the H trace of this line and be tangent to the base of the cone. The V trace will pass through the V trace of the line. If this point cannot be found conveniently, an auxiliary line may be used as in Problem 54.

Problem 56.—*To pass a plane tangent to a given cone and parallel to a given line.*

A line through the vertex of the cone and parallel to the given line determines the required plane as in Problem 55.

NOTE.—The methods of the preceding problems apply equally well to oblique cones and to cones of revolution.

Problem 57.—*To pass a plane tangent to a given cylinder.*

Considering the cylinder as a special case of the cone, that is having its vertex at infinity, the methods of the three preceding problems will apply as well to the corresponding cases of the cylinder.

Problem 58.—*To pass a plane tangent to a given surface of revolution and containing a given point on that surface.*

This problem may be solved by a number of different methods and the choice of method depends on the nature of the surface. A perfectly general method is to construct a cone of revolution tangent to the surface in a circle passing through the given point. The required plane will be tangent to the cone at the element passing through the given point.

Another method which does not completely determine the tangent plane, but is helpful in some cases, is to pass a plane through the given point perpendicular to the

axis of the surface of revolution. This plane cuts a circle from the surface and the given point lies on the circumference of this circle. In the plane of the circle draw a line tangent to the circle at the given point. The tangent plane must contain this tangent line. Special methods may be used in special cases. Planes tangent to cones of revolution can be found as in Problems 54, 55 and 56; to cylinders of revolution as in Problem 57.

Problem 59.—*To pass a plane tangent to a given sphere at a given point on its surface.*

Draw the radius of the sphere to the given point. The required plane is perpendicular to this line and contains the given point.

Problem 60.—*To pass a plane through a given line and tangent to a given sphere.*

First Method.—Pass a plane through the center of the sphere and perpendicular to the given line. This will cut a great circle from the sphere and a point from the given line. Revolve the plane into H or V, and locate the point and circle. From the point draw a tangent to the circle. Counter revolve and find the projections of the tangent line. The plane of this line and the given line is the one required.

Second Method.—From two points on the given line construct cones tangent to the sphere. The intersection of the circles of contact will determine the point of tangency of the required plane. If the axis of one of the tangent cones is taken parallel to H and the other parallel to V, but one ellipse will be required in the construction.

Problem 61.—*To pass a plane tangent to an oblique cone and making a given angle with the plane of the base.*

Construct a cone of revolution with its vertex coinciding with the vertex of the given cone and its elements making the given angle with the plane of the base. The required plane is tangent to both cones.

Problem 62.—*To pass a plane tangent to an oblique cylinder and making a given angle with the plane of the base.*

Construct a cone of revolution at any convenient place, having its base parallel to the base of the cylinder and its elements making the given angle with the plane of the base. Draw a line through the vertex parallel to an element of the given cylinder. The plane containing this line and tangent to the base of the cone will be parallel to the required plane, which will be found by taking its traces tangent to the corresponding traces of the cylinder and parallel to the traces of the plane found.

Exercises

288. A cone of 2 in. base and 1½ in. altitude lies in contact with both H and V. Show the lines of contact.

289. A cone having a vertex angle of 45° has its axis in X. Show the traces of a plane tangent to the cone at a point on its surface ½ in. from H and 3 in. from the vertex.

290. Find the traces of the plane containing the points *a* and *b* and inclined at 75° to H (Fig. 135).

291. A cone of revolution of vertex angle 60° has a slant height of 2 in. Its base lies in V, with center 1½ in. below H. Show the traces of the plane tangent to the cone at a point on its surface ½ in. from V and 2 in. from H.

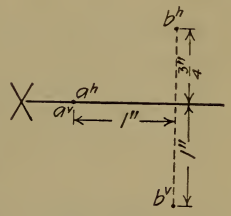


FIG. 135.

292. An oblique cylinder has its axis inclined at 45° to H and parallel to V. Its base is a 1½ in. circle in H. Show the traces of a plane making 60° with H and tangent to the cylinder.

293. A sphere of 2 in. diameter, center in H, ½ in. from V, is tangent to a plane which is inclined at 45° to H and 60° to V. Show projections of the point of tangency.

294. A cone which has a vertex angle of 60° has two planes tangent to it at elements which make 30° with each other. What is the angle between the planes?

295. A sphere 2 in. in diameter, tangent to both H and V, has a great circle cut from it by a plane whose traces meet X at 45°. Show the projections of a line tangent to this circle, the H projection of the line being inclined at 30° to X.

296. Show the projections of a cone of revolution tangent to H, V and P. Find also its vertex angle.

297. An anchor ring with its axis perpendicular to H has an outside diameter of 3 in. and an inside diameter of 1 in. A plane tangent to it has its H trace inclined at 45° to X. ϕ equals 60°. Find the point of tangency.

298. A sphere 3 in. in diameter has its center in H, 1 in. from V. Show the traces of a plane tangent to the sphere at a point 2 in. from V and ½ in. from H.

299. Two spheres of 1½ in. and ¾ in. diameter are tangent to both H and V and to each other. Show their projections.

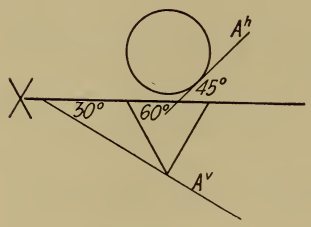


Fig. 136.

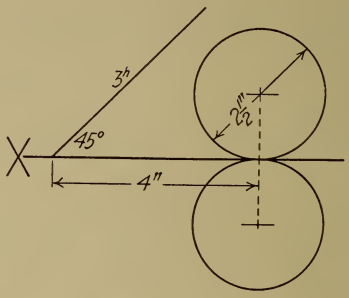


Fig. 137.

300. Show the traces of a plane tangent to both spheres of Exercise 299, and having $\theta = \phi$.

301. The line A, Fig. 136, is the axis of a cylinder which is tangent to the cone. What is the diameter of the cylinder?

302. Find 3^v if the plane 3 is to be tangent to the sphere (Fig. 137)

XVII. Shades and Shadows.

In architectural drawings, and occasionally in other types of drawings, much is added in the way of clearness and beauty by showing shades and shadows, and a single orthographic projection is brought into relief, giving a much clearer idea of the depth of the object in a direction perpendicular to the plane of the drawing.

The direction of the rays of light may be taken in any way desired, but it is almost universal practice to assume

a direction such that both projections of a ray make 45° with X and from over the left shoulder.

In the case of an actual object illuminated by light rays in a certain direction, there are always gradations from extreme high light of illumination to extreme depth of shadow, and further variations are introduced by reflected and diffused light. In mechanical drawings no such gradations are attempted—a surface is either light or dark according as rays of light do or do not reach it.

Surfaces which are on the dark side of an object, from which the light is excluded by the object itself, are said to be in shade. Surfaces which are turned in a direction to receive the light, but from which the light is excluded by other objects or other parts of the same object, are said to be in shadow. On an actual object shadows generally appear darker than shade, but again no such distinction is attempted in mechanical drawings.

When a solid body is interposed between a plane and the source of light, the body casts a shadow on the plane. Light is excluded from the space between the body and its shadow, and this space is called umbra, or invisible shadow, or shadow in space. The outline of the shadow will be determined by a series of light rays tangent to the solid. In other words the shadow outline is the shadow of the linear outline along which light rays are tangent to the solid. This linear outline on the body, which may be plane or otherwise, is called the line of shade, and it is evident that it is the line on the body separating light from shade. Determining the shadow of an object then reduces to the problem of finding the shadow of the line of shade. Since this line of shade consists of a series of points, any problem of finding shadows is merely a problem of finding shadows of points.

The shadow cast on a surface by a point is the point of intersection of a light ray through the point with the

surface. The invisible shadow of a point is a straight line coinciding with this ray of light.

The shadow of a line consists of the shadows of its individual points. In general the shadow of a line is a line, and its invisible shadow is a surface. If the line is straight its invisible shadow is a plane, and its shadow is the line of intersection of the plane with the surface on which the shadow is cast.

The following principles should be kept in mind in plotting shades and shadows, as their application simplifies the work to a considerable extent:

1. The shadow of a straight line on a plane is a straight line.

2. If any line, either straight or plane curve, casts a shadow on a plane to which it is parallel, its shadow is the same length and shape as the line.

3. If a straight line is perpendicular to H or V, then the H or V projection of its shadow on any surface is a straight line at 45° to X.

4. The point where the shadow of one line crosses another line not in the same plane with it can be found by finding shadows of both lines on any convenient plane. From the point of intersection of the two shadows draw a ray of light backward, and where this ray cuts the second line is the required point. If this ray of light is produced backward till it intersects the first line, it gives the point whose shadow falls on the second line.

5. The shadow of a curved line on a plane, or of any line on a curved surface, is determined by finding the shadows of a number of points of the line on the surface and joining the point shadows thus found. The points must be taken sufficiently close together to determine the required shadow with the desired degree of accuracy.

6. The general method of finding the shadow of an object will be first, determine the line of shade; second find its

shadow. It is sometimes more convenient to reverse this procedure, first finding the shadow and then by drawing rays of light back from the shadow, determining the line of shade.

Projecting Planes and Shadow Planes

The object is considered as being in the third quadrant, and plan and elevation are drawn. The planes on which shadows are cast are taken parallel to H and V. The horizontal shadow plane is taken as the plane on which the object rests, and is called the ground plane, and will be referred to as G. P. The vertical shadow plane is taken behind the object at a distance such that part of the shadow will fall on it. This plane will be referred to as V. S. P.

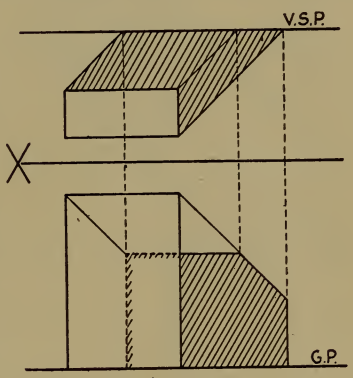


Fig. 138.

Only that part of the shadow on V. S. P. which comes above G. P. need be found, and also only that part on G. P. which comes in front of V. S. P. It is sometimes necessary to find a point or two behind V. S. P. or below G. P. in order to determine the shadow. The two shadows will join along the line of intersection of the two shadow planes.

In Fig. 138 are shown the shadows of a rectangular block. The method of locating the points is shown by the construction lines. The hidden portion of the shadow on V. S. P. may be shown by dotted lines as indicated, or may be omitted entirely. The shades and shadow of cylinder are shown in Fig. 139 and of a cone in Fig. 140. The lines of shade of the cone are best determined by first finding the shadow. The shadow of a cone on the plane of its base is

determined by finding the shadow of its vertex and drawing tangent lines from this point to the base. The points of tangency determine the elements which are the lines of shade.

For finding the lines of shade on a solid of revolution with its axis vertical, and its shadow on the G. P. the following method is useful. Cut the solid by a series of planes perpendicular to the axis of the solid, cutting the solid in

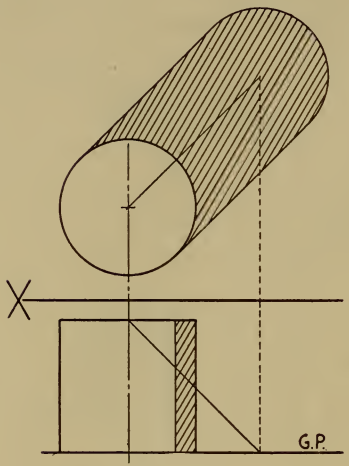


Fig. 139.

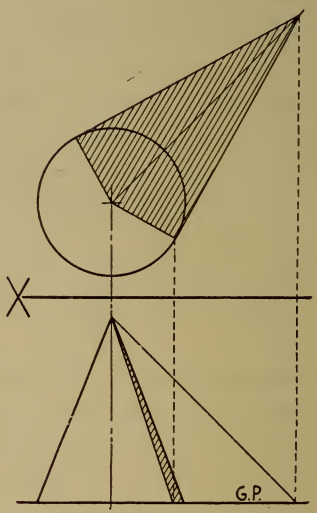


Fig. 140.

circles. Find the shadows of these circles on the ground plane (see principle 2). The curve enveloping these circular shadows is the required shadow. The line of shade can be found by drawing light rays from the points where the circular shadows are tangent to the envelope back to the original plane sections, each section giving two points on the line of shade.

Pilet's Method. This is a method of finding shadows of circles on surfaces of revolution, and is useful in a number of problems. It depends on two principles, one of which

has already been stated as Principle 4. The other is that the shadow of a horizontal circle on a vertical plane inclined at 45° to V shows as a circle in the V projection, the diameter of the shadow projection being to the diameter of the circle as the half diagonal of a square is to a side of the square. This is illustrated in Fig. 141, in which *abcd* is a square circumscribed about the horizontal circle, and *a'b'c'd'* is the V projection of its shadow on the plane *3*.

To illustrate the application of these principles in Pilet's

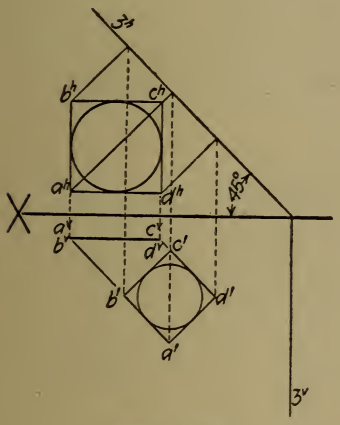


Fig. 141.

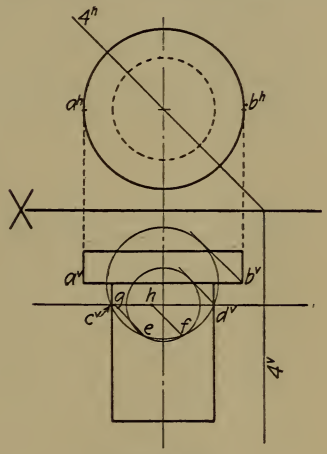


Fig. 142.

method, let us consider a cylindrical column with a circular lintel, as in Fig. 142, required to find the shadow of the lintel on the column. The lower circle *ab* is the line of shade. Assume a vertical 45° plane through the common axis of column and lintel, and find the V projection of the shadow of circle *ab* on this plane. This will be a circle with its center on $a^v b^v$ and with diameter equal to the semi-diagonal of a square of side *ab*. Take any horizontal section of the column, as *cd*, and find the V projection of the shadow of this circle on the vertical plane. This will be a circle with center on $c^v d^v$. The intersection of the two

shadow circles gives two points *e* and *f*, which must be the shadows of the points in which the shadow of the circle *ab* cuts the circle *cd*. In other words, *e* and *f* are the shadows on the 45° plane of two points on the required curve. Draw light rays back, cutting $c^v d^v$ in *g* and *h*, which are two points on the required curve. By taking a number of planes such as *cd*, any desired number of points can be determined and the required curve obtained.

For further methods in shades and shadows the student is referred to a series of articles by Prof. A. D. F. Hamlin, in the American Architect, beginning in 1889. These articles constitute a most excellent treatise on the subject, and from them was taken the Pilet's method explained above.

Exercises

303. Find shadow of lintel on column (Fig. 143).

304. Find shade and shadow of lintel on column (Fig. 144).

305. Find shade and shadow of lintel on column (Fig. 145).

306. Find shades and shadow of lintel on column (Fig. 146).

307. Find shade and shadow of lintel on column (Fig. 147).

308. Find shades and shadow of lintel on column (Fig. 148).

309. Find shades and shadows (Fig. 149). V. S. P. as shown.

310. Find shades and shadows (Fig. 150). V. S. P. as shown.

311. Find shades and shadows (Fig. 151). No V. S. P.

312. Find shades and shadows (Fig. 152). V. S. P. as shown.

313. Find shades and shadows (Fig. 153). No V. S. P.

314. A cylinder has as its base a 2 in. circle the plane of which is parallel to H. The cylinder is inclined at 60° to H and is parallel to V. Find shades and shadows. No V. S. P.

315. Find shades and shadows (Fig. 154). No V. S. P.

316. Find shades and shadows (Fig. 155). No V. S. P.

317. Find shades and shadows (Fig. 156). No V. S. P. Each block is 2 in. × 1 in. × ½ in., and the upper one is inclined at 30° to H.

318. Find shades and shadows (Fig. 157). V. S. P. as shown.

319. Find shades and shadows (Fig. 158). No V. S. P.

320. Find shades and shadows (Fig. 159). V. S. P. as shown.

321. Find shades and shadows (Fig. 160). No V. S. P.

322. Find shade and shadow on the ground plane of a 2 in. sphere.

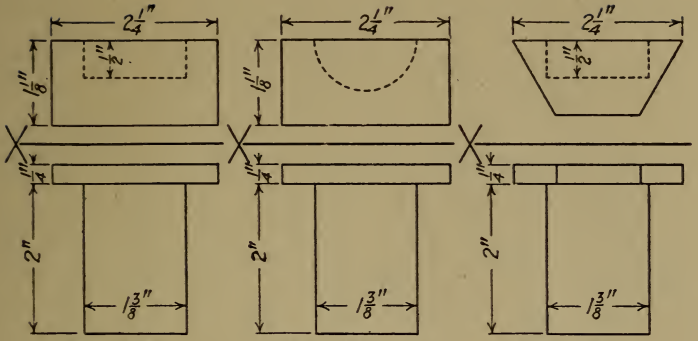


FIG. 143. FIG. 144. FIG. 145.

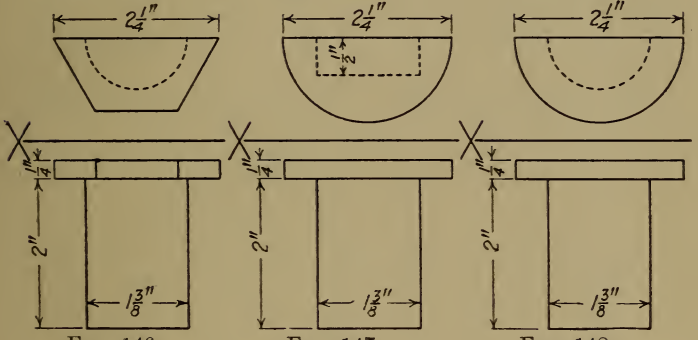


FIG. 146. FIG. 147. FIG. 148.

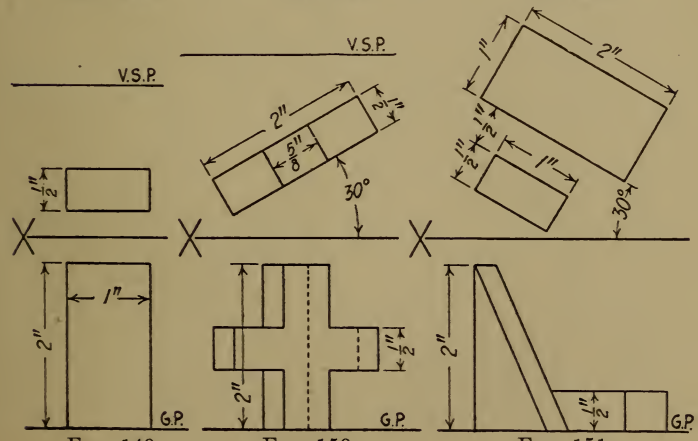


FIG. 149. FIG. 150. FIG. 151.

FIGS. 143 to 151.

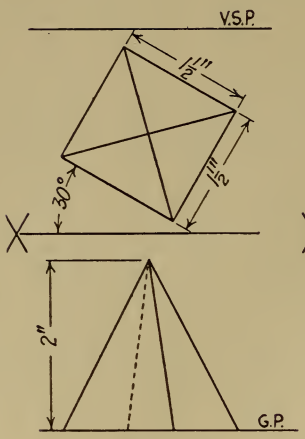


FIG. 152.

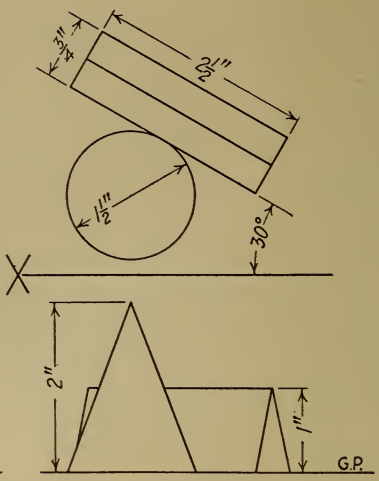
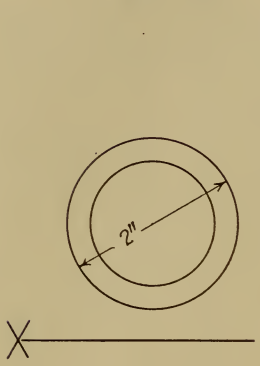
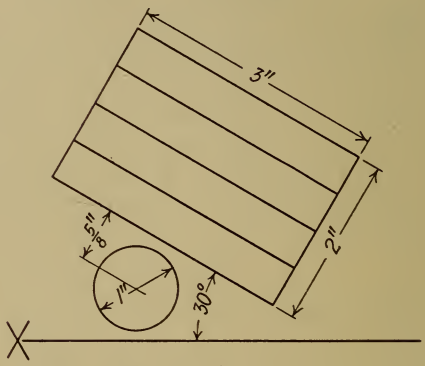


FIG. 153.





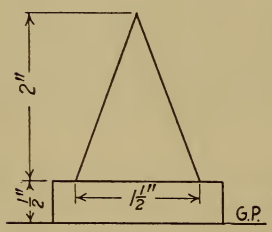


FIG. 154.

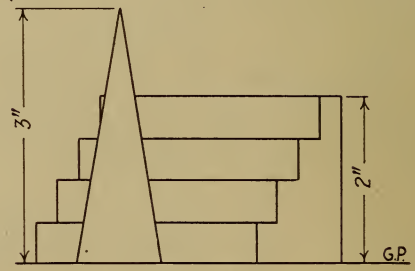


FIG. 155.

FIGS. 152 to 155.

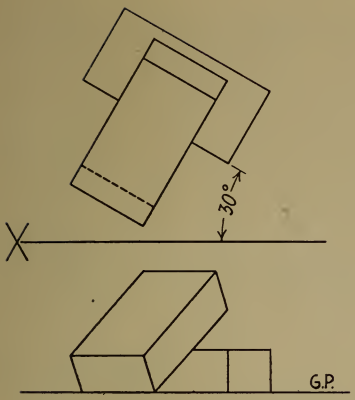


Fig. 156.

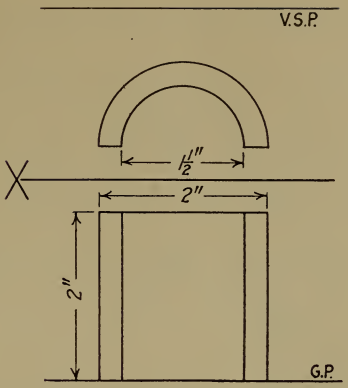


Fig. 157.

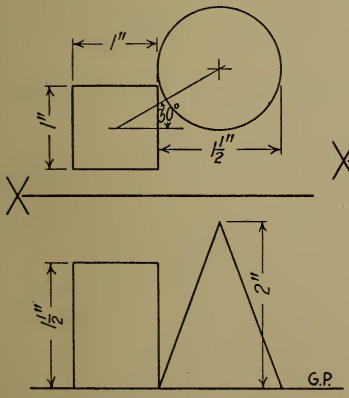


Fig. 158.

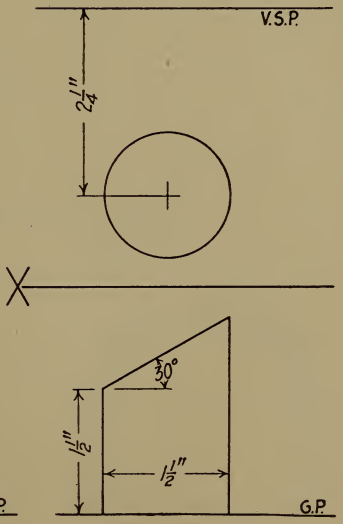


Fig. 159.

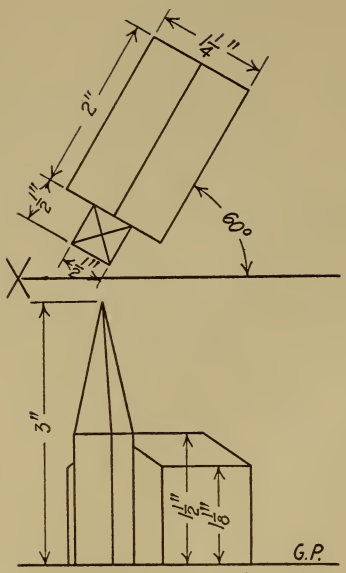


Fig. 160.

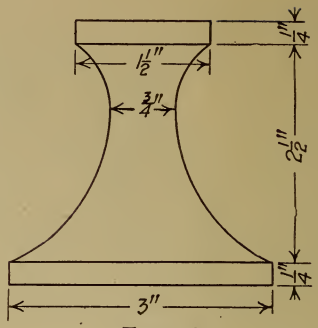


Fig. 161.

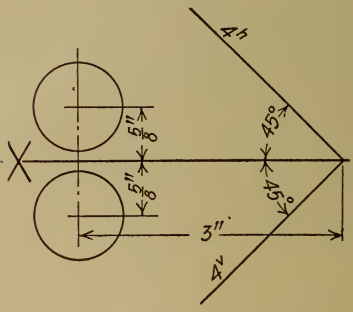


Fig. 162.

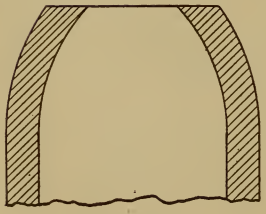


Fig. 163.

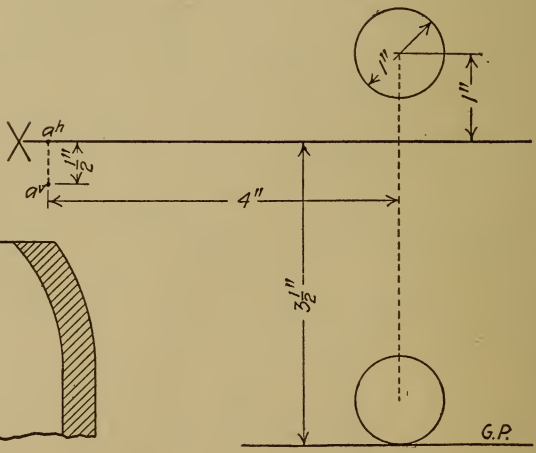


Fig. 164.

Figs. 160 to 164.

323. Find shades and shadow on the ground plane of the solid of revolution shown in Fig. 161.

324. Show the projections of the shadow on plane 4 of the sphere shown in Fig. 162, also the shade on the sphere.

325. Show projection of the light spot in the dome (Fig. 163).

326. A luminous point 1 in. above the vertex of a cone is 1½ in. from the axis of the cone. The cone has a base 2 in. in diameter and an altitude of 2 in. Find the lines of shade on the cone and its shadow on the plane of the base.

327. With *a* as source of light, find shades and shadows in Fig. 164. No V. S. P.

XVIII. Sections.

When a solid is divided by a plane the surface of division is called a section. If in a prism the cutting plane contains an edge, in a cylinder an element or in a cone the axis, the section is called longitudinal. In a solid of revolution the longitudinal section becomes the meridian section. Since the cutting plane contains the axis and a generating element, all meridian sections of a solid of revolution are equal. All planes perpendicular to the axis will cut circles called parallels, of which the smallest, if greater than zero, is called the circle of the gorge, and the largest the circle of the equator.

The general method of finding the section by an oblique plane is to cut both the solid and the plane by auxiliary planes so assumed as to cut the simplest lines, that is the lines most easily found, from each. Since the required line of intersection is common to both the secant plane and the surface of the solid, the lines cut from them by an auxiliary plane will intersect in points of the required line.

In order to get the simplest lines the auxiliary planes should in general be taken as follows:

(*a*) In solids bounded by plane surfaces, coincident with the surfaces or through edges.

(*b*) In oblique cylinders, parallel to the axis.

(*c*) In oblique cones, passing through the vertex.

(*d*) In solids of revolution, perpendicular to the axis.

Problem 63.—*To show the projections and true form of the section of a polyhedron.*

The section is a right lined figure which can be determined by finding the vertices from the intersections of the edges with the cutting plane, or the sides of the figure can be determined by finding the intersections of the planes of the solid with the cutting plane.

The true form of the section can be found by revolving the plane of the section into H or V.

In case the position of the solid can be assumed so as to

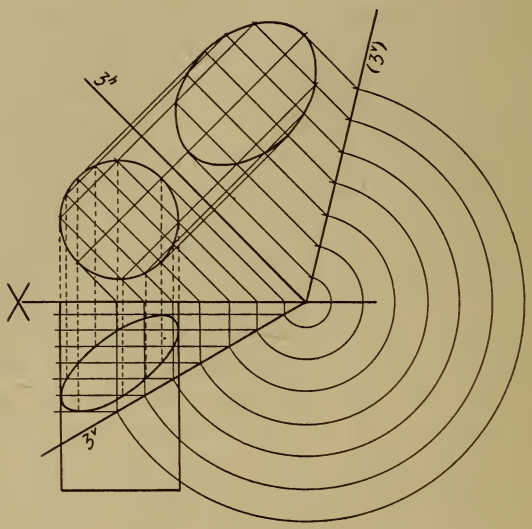


Fig. 165.

bring the cutting plane perpendicular to H or V the work is somewhat simplified. One projection of the required section is then a straight line coinciding with the oblique trace of the cutting plane. From this the vertices of the other projection can readily be projected.

Problem 64.—*To show the projections and true form of the section of a cylinder.*

The projections of the section are found by drawing elements of the cylinder and finding their points of intersection with the cutting plane. Enough points should be found to properly determine the outline of the section, which will be a smooth curve joining these points.

In the case of a cylinder of revolution the section, not considering the bases of the cylinder, will be two parallel lines, a circle or an ellipse, according as the cutting plane is parallel, perpendicular or oblique to the axis of the cylinder. The projections of the section may befound by drawing a number of auxiliary planes perpendicular to the axis of the cylinder, as illustrated in Fig. 165. Let it be required to find the section of the cylinder of revolution by plane *3*. The axis of the cylinder being vertical, a series of horizontal planes is drawn, each plane giving two points on the required curve.

The true form of the section is found by revolving *3* into H about 3^h as axis. If the true form only is required, the lengths of the major and minor axes may be determined and the ellipse constructed by any convenient method.

Problem 65.—*To show the projections and true form of the section of a cone.*

The method of finding the projections and true form of the section will be the same as in the case of the cylinder.

In the case of a cone of revolution the section, not considering the base, is bounded by two intersecting straight lines if the cutting plane passes through the vertex. Otherwise the section is a circle if the cutting plane is perpendicular to the axis of the cone, an ellipse if oblique to the axis but making a greater angle with it than the elements of the cone, a parabola if the same angle as the elements, and an hyperbola if a less angle.

Problem 66.—*To show the projections and true form of the section of a solid of revolution.*

Place the solid with its axis perpendicular to H or V.

Cut it and the secant plane by auxiliary planes taken perpendicular to the axis of the solid, cutting circles from the solid and straight lines from the plane, which intersect in points of the required line. The true form is found as before. This method is the one used in Fig. 165, and may be applied to any solid of revolution.

Exercises

328. A hexagonal prism of 1 in. side and 3 in. long has a hole ¾ in. in diameter, the axis of which coincides with the axis of the prism. Show the section by a plane making 60° with the axis of the prism.

329. A pentagonal prism of 1 in. side and 4 in. long is cut by a plane bisecting the axis and making 45° with it. Show true form of section.

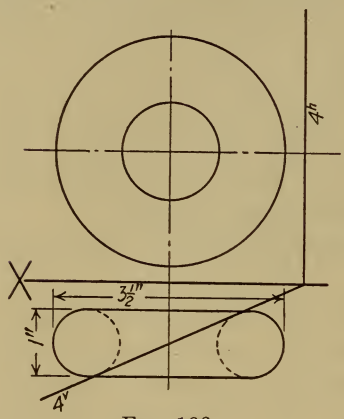


Fig. 166.

330. Show the true form of the section of a cube by a plane passing through its center and perpendicular to a diagonal.

331. A cone having a vertex angle of 90° has a parabola cut from it by a plane which is ¾ in. from the vertex of the cone. Show the true form of the section.

332. A cone of 3 in. altitude and base 2 in. in diameter has an hyperbola cut from it by a plane ½ in. from the axis of the cone. Show the true form of the section.

333. A cylinder of 2 in. diameter with its axis vertical is cut by a plane whose traces are inclined at 30° to X. Show projections and true form of the section.

334. A cone of revolution has a vertex angle of 60°. Show the traces of a plane which will cut the cone in an ellipse whose major axis is 2 in. and minor axis 1 in.

Note—See text on page 101.

335. A cone of revolution of vertex angle 60° is cut by a plane at 60° to its axis, 1½ in. from the vertex. Show the projections of a cylinder of revolution of which the ellipse is an oblique section.

336. The axis of a cylinder of 1½ in. diameter is inclined at 45° to

H and 30° to V. The cylinder is tangent to X. Show its intersections with H and V.

337. A sphere of 2 in. diameter tangent to both H and V is cut by a plane whose traces meet X at 45° and pass through the projections of the center. Show projections and true form of the section.

338. Show projections and true form of the section of the torus cut by plane *4* (Fig. 166).

339. A solid of revolution is generated by revolving a 2 in. circle about a cord ½ in. from the center. Show true form of section cut by a plane passing through the center of the equator and inclined at 45° to the axis of the solid.

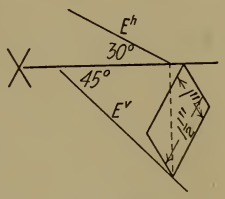


Fig. 167.

340. A cone of vertex angle of 60° has its vertex 1 in. from H and its axis inclined at 45° to H. Show its intersection with H.

341. A stick of octagonal section, the circumscribing circle being 1½ in. in diameter, has its axis perpendicular to V. It is cut by a vertical plane making 45° with V. A second stick with its axis parallel to V and inclined at 30° to H exactly meets the section of the first stick. Show the right section of the second stick.

342. A stick, the section of which is a parallelogram, has its V trace and the projection of an edge shown in Fig. 167. What is the angle between the two faces meeting in edge *E*?

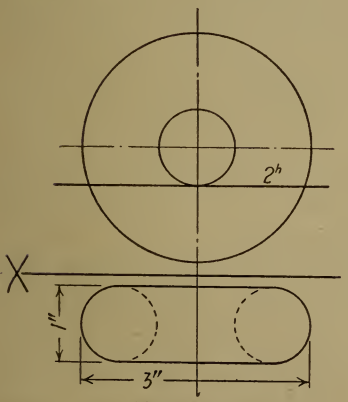


Fig. 168.

343. Show the section of a regular tetrahedron cut by a plane passing through its center and parallel to two adjacent edges.

344. A solid of revolution is formed by revolving a circular arc of 90° about its chord which is 3 in. long. Show the true form of the section of the solid by a plane parallel to the axis and ¼ in. from it.

345. Show the true form of the section cut from the torus of Fig. 168 by the plane *2*.

Note.—This section is the Lemniscate of Bernouilli.

346. A cone of revolution of vertex angle 60° with its vertex lying

in H, 1½ in. from V, and its axis perpendicular to H, is cut by a plane whose H trace makes 45° with X and V trace 30°. The V trace of the plane is 1½ in. from the V projection of the vertex. Show projections and true form of the section.

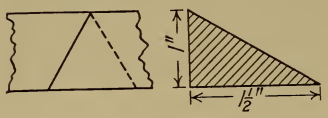


Fig. 169.

347. The stick shown in plan and end elevation in Fig. 169 is cut by a plane such that the plan of the section is an equilateral triangle. Show the true form of the section.

XIX. Intersection of Surfaces.

If a given surface made up of planes intersect a second surface the line of intersection will be made up of parts of the sections cut by the planes making up the first surface, and may be determined by the methods given under Sections.

In general, to find the line of intersection of two surfaces auxiliary planes are taken cutting lines from the given surfaces which intersect in points of the required line. The amount of work required to find any desired line of intersection will depend largely on the judgment used in selecting auxiliary planes so as to cut simple lines from the given surfaces.

Problem 67.—*To find the projections of the line of intersection of two cylinders.*

Take auxiliary planes parallel to the axes of both cylinders, cutting straight line elements from each. These elements intersect in points of the required curve.

In Fig. 170, given two oblique cylinders whose axes are the lines A and B, the base of one being taken in H, the other in V. The first step is to determine the direction of the auxiliary planes. Through any point in space, such as e, draw lines A_1 and B_1 parallel to the axes of the given cylinders, and find the plane of these lines, plane *3*. Any auxiliary plane parallel to *3* will cut straight line elements from both cylinders, and these elements intersect

in four points of the required line. One such auxiliary plane is shown as plane *4*, cutting lines *C* and *D* from one cylinder and *E* and *F* from the other. These lines intersect in points *a*, *b*, *c* and *d*, which are points on the required curve. A sufficient number of such planes must be used to give points enough to determine the curve.

The foregoing method is perfectly general, and is much

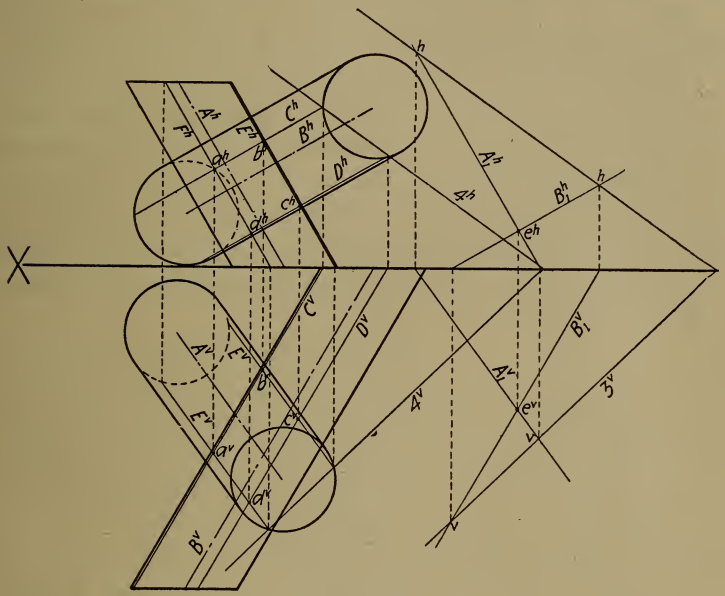


Fig. 170.

simpler in the case of cylinders of revolution. In this case one cylinder can be placed with its axis perpendicular to V, the other parallel to H, and the auxiliary planes are taken parallel to H.

Problem 68.—*To draw the projections of the line of intersections of a cylinder and a cone.*

The general method is the same as in the preceding problem. The auxiliary planes, in order to cut straight lines from both cylinder and cone, must pass through the

vertex of the cone and be parallel to the axis of the cylinder. Hence if a line parallel to the axis of the cylinder is drawn through the vertex of the cone, all auxiliary planes must have their traces containing the corresponding traces of this line.

Problem 69.—*To draw the projections of the line of intersection of two cones.*

Again the same general method is used, the auxiliary planes passing through the vertices of both cones. Hence, if a line is drawn joining the two vertices, all auxiliary planes must have their traces containing the corresponding traces of this line.

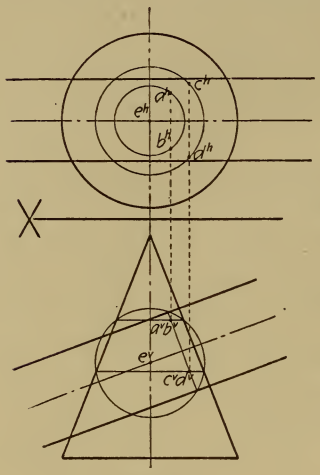


Fig. 171.

Problem 70.—*To draw the projections of the line of intersection of a sphere and a polyhedron.*

A series of planes parallel to H or V will cut circles from the sphere and polygons from the polyhedron, which will intersect in points of the required line of intersection.

Problem 71.—*To draw the projections of the line of intersection of two surfaces of revolution, axes intersecting.*

Place the surfaces with axes parallel to V and one of the axes perpendicular to H. With center at the intersection of the axes take a series of spheres which intersect the given surfaces in circles, the elevations of which are straight lines. The plan of one is a circle and of the other an ellipse, but the points of intersection can be determined without the use of the ellipse.

This method as applied to the case of a cylinder and cone of revolution is illustrated in Fig. 171. With center at *e*

the point of intersection of the two axes, draw a sphere which cuts both cylinder and cone in circles. The V projection of both circles are straight lines, the H projection of one is the circle shown, and of the other an ellipse, which is not shown. The points *a*, *b*, *c* and *d* are determined from the V projections and the circular H projection. A sufficient number of spheres must be taken to give points enough to determine the curve.

Problem 72.—*To draw the projections of the line of intersection of two surfaces of revolution, axes not intersecting.*

Place the surfaces as before, one axis perpendicular to H and the other parallel to V. Cut the surfaces by horizontal planes. The elevations of the sections are straight lines, the plan of one a circle, of the other a curve which will generally have to be plotted by points, but only a small part of the curve need be drawn for each section.

Exercises

348. *A* is the axis of a cylindrical hole 1¾ in. in diameter. Show the projections of the line of intersection (Fig. 172).

349. A pyramid with an altitude of 3 in. has a base 2 in. square, parallel to H with its diagonal parallel to V. It is intersected by a triangular hole of 1 in. side, the axis of which is 1 in. above the base, ¾ in. from the axis of the pyramid and inclined at 30° to V. Show plan and elevation of the pyramid .

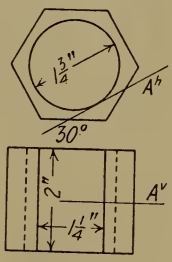


Fig. 172.

350. A vertical tube of 2½ in. outside diameter and 1½ in. inside diameter has a horizontal cylindrical hole bored through it of 1½ in. diameter. The distance between the axes is ⅛ in. Show the projection of the line of intersection on a vertical plane to which the axis of the horizontal hole is inclined at 45°.

351. Show the projections of the points in which the line *A* pierces the sphere (Fig. 173).

352. A sphere of 3 in. diameter is intersected by a cylinder of 2 in. diameter. The axis of the cylinder is ½ in. from the center of the

sphere. Show H and V projections of the line of intersection when the axis of the cylinder is parallel to H, inclined at 30° with V and in a horizontal plane with the center of the sphere.

353. Show H, V and P projections of the line of intersection of the cylinders (Fig. 174).

354. Show the projections of the lines of intersection of the solid

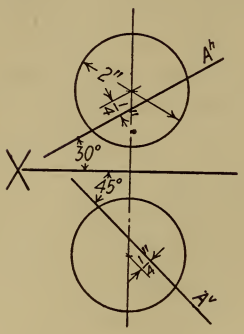


Fig. 173.

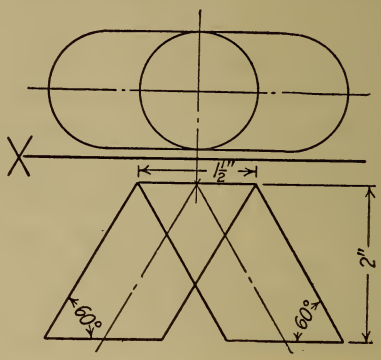


Fig. 174.

of revolution of Exercise 344 with a sphere of 1½ in. diameter, center of sphere on axis of solid and ½ in. from equatorial plane.

355. Show the projections of the lines of intersection of a cylinder and cone of revolution. The vertex angle of the cone is 60°, altitude 2½ in. The diameter of the cylinder is 1 in., and the axis of the cylinder intersects the axis of the cone at a point 1¼ in. from the vertex, and at an angle of 75°.

356. A surface generated by the revolution of a circular arc of 120° about its chord, which is 3 in. long, is intersected by a cylinder of 1½ in. diameter, the axes meeting at the middle point at an angle of 45°. Show the projections of the lines of intersection.

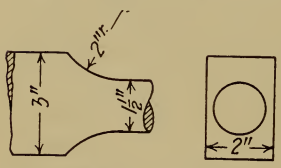


Fig. 175.

357. A connecting rod stub end of rectangular section is finished in a lathe to the outline indicated in Fig. 175. Show complete plan and elevation.

358. An anchor ring generated by a 1 in. circle revolving about an axis ¾ in. from its center is intersected by a cylinder of 1 in. diameter, the axis of the cylinder meeting the axis of the surface at 30° and

tangent to the inner surface of the ring. Show the projections of the line of intersection.

359. Show the line of intersection of the cone and cylinder of Fig. 176.

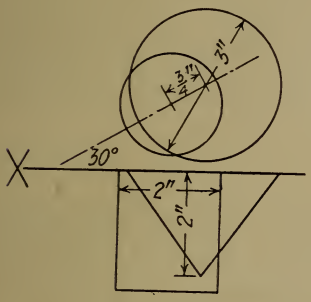


Fig. 176.

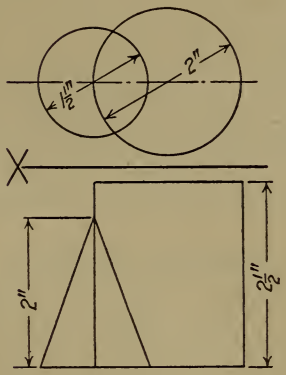


Fig. 177.

360. Show the line of intersection of the cone and cylinder of Fig. 177.

361. Show the projections of the line of intersection of the torus of Exercise 345 with a cylinder of 2 in. diameter, axis of cylinder parallel to axis of torus and 1 in. from it.

362. Two cylinders of revolution of 2 in. diameter intersect with the axis of each cylinder tangent to the other cylinder. One has its axis perpendicular to H, the other parallel to X. Show the projections of the curve of intersection.

363. Points *a*, *b* and *c* form an equilateral triangle of 2½ in. side lying in H. A point *p* is 1½ in. from *a*, 2 in. from *b* and 2½ in. from *c*. Show the projections of the point.

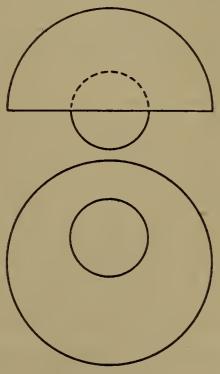


Fig. 178.

364. Two spheres of 2½ in. and 3 in. diameters have the plans of their centers 1⅛ in. apart on a line making 30° with X. The center of the smaller sphere is 1¼ in. above H, of the larger in H. Find the traces of the plane, the projections of the center and the radius of the circle in which the spheres intersect.

365. A right cone of slant height 3 in., diameter of base 2½ in., lies against H with the plan of the axis parallel to X. It is intersected by a cylinder of 1½ in. diameter lying against H and inclined at 60° to V. The plan of the axis of the cylinder crosses the plan of the axis of the cone at a point 2 in. from its vertex. Show projections of the line of intersection.

366. A sphere of 1½ in. diameter has its center ½ in. above the center of a hemispherical shell of 4 in. diameter, as shown in Fig. 178. Find the shadow of the sphere on the concave surface of the shell.

367. *abc* is a triangle in H. *ab* is 3 in., *bc* is 4 in. and *ac* is 4½ in. *d* is a point in space. θ for the line $\overline{ad}$ is 30°, for $\overline{bd}$ 45° and for $\overline{cd}$ 45°. Show the projections of the point *d*.

XX. Helicoidal Surfaces.

A helix is the curve traced upon a cylinder of revolution by a point having uniform motion of rotation about the cylinder and at the same time a uniform motion of translation along the elements of the cylinder. The curve therefore makes equal angles with the elements and when the surface of the cylinder is developed forms a straight line which is inclined to the developed elements at an angle whose tangent equals the circumference of the cylinder divided by the axial pitch of the helix. The axial pitch is the distance from one coil of the helix to the next, measured along an element, as in Fig. 179.

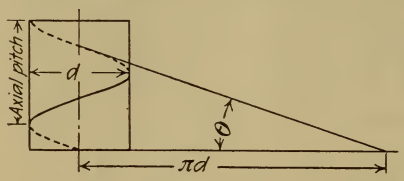


Fig. 179.

The surface generated by a line moving in contact with a given helix and in some fixed relation to its axis, is called a helicoidal surface. Ordinary screw surfaces are helicoidal surfaces in which the generating line intersects the axis; if at right angles the surface is part of the square thread; if at 60°, the ordinary form of V thread.

If a helicoidal surface is intersected by cylinders of revolution having the same axis, the lines of intersection

are helices having the same axial pitch, but inclined to the base of the cylinder at angles whose tangents vary inversely as the diameter of the cylinder.

In general helicoidal surfaces are not developable, and are sometimes called "skew screw surfaces."

A single exception occurs in the case when a line moves so as to be in all positions tangent to the given helix. The surface thus generated can be developed, because consecu-

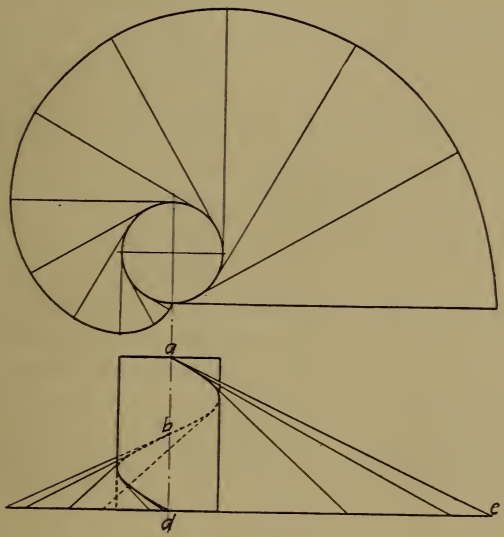


FIG. 180.

tive elements are consecutive tangents to the helix, and therefore intersecting lines. This surface is called the developable helicoid.

Figure 180 represents that portion of the surface included between two parallel planes at a distance apart equal to the pitch of the helix. If a card be cut in the form of the right triangle *adc*, having *ad* equal to the axial pitch of the helix and *dc* equal to the circumference of the cylinder of the helix, and if this card be placed in the position indicated and rolled

about the cylinder in contact with it, the line *ac* will be in all positions in contact with the helix *abd* and will therefore generate the desired surface. Two like surfaces will be generated meeting at the helix in a cuspidal edge. The point *a* will trace on the upper plane an involute of the plan of the helix, while the point *b* will trace the opposite involute on the lower plane. Only the lower surface is shown in the figure.

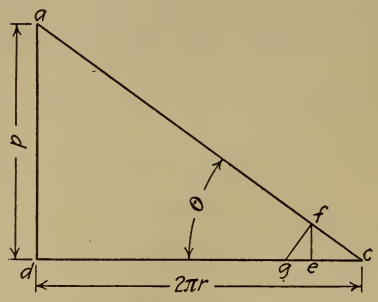


Fig. 181.

It can be shown that when the helicoidal surface is developed, the cuspidal edge will develop into a circular arc of radius $R = r \sec^2\theta$, in which r is the radius of the cylinder and θ the angle which an element makes with the plane of the base. The value of R can be found graphically by the triangle construction shown in Fig. 181. Lay off *ce* equal to r, draw *ef* perpendicular to *cd* and *fg* perpendicular to *ca*, then *cg* is equal to R. The line of intersection of the surface with a plane perpendicular to the axis will develop into an involute of the arc of radius R.

Exercises

368. A helicoidal surface is generated by a horizontal line in contact with a vertical axis and a helix of 2 in. pitch. Show the projections of the surface included between two horizontal planes 2 in. apart and two concentric cylinders of 1 in. and 2 in. diameters, axes coinciding with the axis of the helix.

369. A screw is 2 in. outside diameter and has two square threads per inch. Show the projections of the portion included between two parallel planes 2 in. apart.

370. A V thread of 1 in. pitch and 4 in. outside diameter is cut by a plane parallel to the axis and 1 in. from it. Show the section.

371. Show the section of the V thread of the preceding exercise by a plane perpendicular to the axis.

372. A cylinder 1¼ in. in diameter and 2 in. long has a helix of 2 in. pitch drawn on its surface. Show the projections of the surface generated by a line moving so as to be always tangent to the helix.

373. A screw conveyor is of the form of a developable helicoid. The diameter of the inner helix is 12 in., of the outer one 30 in. and the pitch is 36 in. Show projections and development of one coil. Scale, 1 in. = 1 ft. 0 in.

374. A pulley of 2 ft. 0 in. diameter, 2 ft. 4 in. above a floor, is connected by a crossed belt 8 in. wide to a pulley of 2 ft. 0 in. diameter and 1 ft. 2 in. below the floor. The pulleys are 4 ft. 0 in. apart, center to center. Assuming that the belt in twisting takes the form of a helicoidal surface, show the lines of intersection with the floor. Scale, 1 in. = 1 ft. 0 in.

XXI. Hyperboloid of Revolution of One Nappe.

This surface may be generated (1) by the rotation of a straight line about an axis not in the same plane, (2) by the rotation of an hyperbola about its conjugate axis, (3) by a circle of variable radius, the center of which moves along the conjugate axis of an hyperbola, the plane of the circle being perpendicular to this axis and the circumference of the circle always in contact with the hyperbola. The surface is best studied by considering it as generated by the first method.

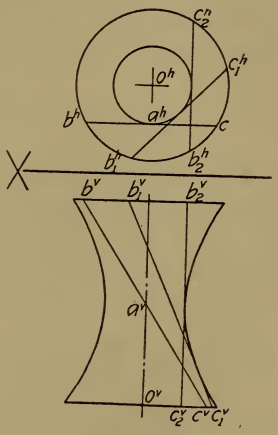


Fig. 182.

Suppose the line $\overline{bc}$, Fig. 182, to rotate about the vertical axis O, keeping at a constant distance Oa from it and making a constant angle θ with H. The line will take the successive positions indicated, the plan being tangent to the circle Oa, from which the elevations are determined.

The surface thus generated will have circles for its cross sections, the smallest being the gorge circle of radius Oa.

Any meridian section will be an hyperbola having O as its conjugate axis. It will be seen that the same surface would have been generated if the line having the plan $b^h c^h$ had had the point c in the plane of the top and b in the plane of the lower base. Hence the surface is a doubly generated one, being made up of two sets of elements. It is the only surface of revolution which is doubly generated. Any point on the surface will lie on two straight line elements which are, however, not of the same set. These two elements determine the plane tangent to the surface at this point.

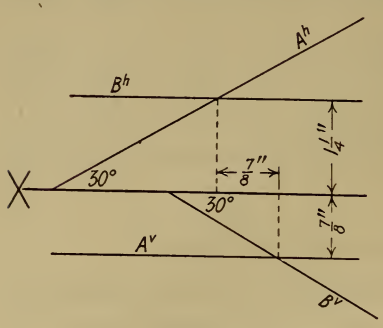


Fig. 183.

From the nature of the generation of this surface no two elements of the same set can intersect or be parallel, hence the surface is not developable.

Exercises

375. Show the projections of an hyperboloid of revolution having the radius of the gorge circle 1 in. and the elements inclined at 30° to the planes of the bases, which are 2 in. apart. Show also the lines of intersection by planes parallel to V and at distances of ¾ in. and 1¼ in. from the axis.

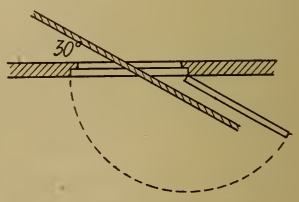


Fig. 184.

376. A point on the surface of the hyperboloid of the preceding exercise lies in a meridian plane at 60° to V and is 1½ in. from the axis of the hyperboloid. Show the traces of the plane tangent at this point.

377. An hyperboloid is generated by revolving line B about A as axis (Fig. 183). Show the projections of another line which, when revolved about A, will generate the same surface.

378. A rope passes through a doorway as shown in plan in Fig. 184.

The door is 3 ft. 0 in. wide and the rope is inclined at 45° to the horizontal. Show where the door must be cut to allow it to be shut.

379. A surface generated by rotating a cube of 2 in. edge about a diagonal, is cut by a plane parallel to the axis and 1 in. from it. Show the outline of the section.

380. A square prism of 1 in. side is cut by a plane so that the section has angles of 60° and 120° and the long diagonal measures 3 in. Show the section.

XXII. Practical Applications.

Gearing.—In the transmission of power by toothed gears the motion transmitted from one shaft to another is the same as that which would be transmitted by rolling of one surface on another. These surfaces, which are imaginary on an actual gear, are called the pitch surfaces. In the most common case, that of spur gears, which are used when the two shafts are parallel, the pitch surfaces are cylinders. In special cases these cylinders may have right sections other than circular, as in the case of elliptical gearing.

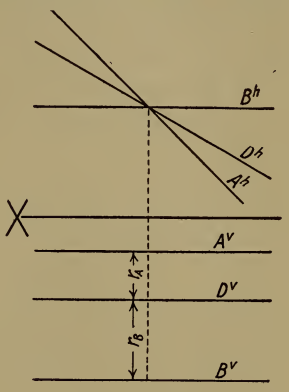


Fig. 185.

Two particular types of gears involve the principles of Descriptive Geometry in their design. Skew bevel gears used when the two shafts are neither parallel nor intersecting, have pitch surfaces which are hyperboloids of revolution.

Suppose that A and B, Fig. 185, are the center lines of the two shafts that are to be connected by skew bevel gears. If some third line intermediate between A and B, such as D, is revolved in turn about A and B, it will generate two hyperboloids of revolution which will be in contact along the line D, and which will roll on each other as they are

rotated about the axes A and B. These hyperboloids therefore constitute the pitch surfaces of the required gears. The speed ratio of the two gears will depend on the position of the line D. The angles between the H projections D^h and A^h, and between D^h and B^h, may be called the skew angles of the shafts A and B. If the radii of the gorge circles of the hyperboloids are made in the ratio of the tangents of the skew angles, then the speed ratio will be the ratio of the sines of these angles. The problem will usually be in the following form: Given the two shafts and the required speed ratio, to determine the element of contact and the pitch surfaces.

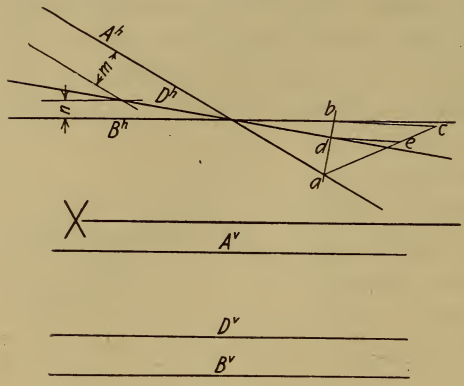


Fig. 186.

The following method by George B. Grant, M. E., is a direct solution.[1]

Suppose A and B, Fig. 186, to be the given shafts, and the required speed ratio $m : n$. Draw lines parallel to A^h and B^h at distances from them whose ratio is $m : n$. The point of intersection of these lines will be a point on D^h, as will also the point of intersection of A^h and B^h. The sum of the two gorge radii is the distance between A^v and B^v, and this is to be divided in the ratio of the tangents of the skew angles. Draw any line ab perpendicular to D^h and through a draw a line ac at any convenient angle. Lay off ac equal to the distance from A^v to B^v, connect bc and draw de parallel to bc. ae and ec will then be the required gorge radii. D^v is thus determined at a distance ae below

[1] For a complete discussion of this subject, see Mr. Grant's book "Gearing."

A^v. The hyperboloid pitch surfaces can then be drawn by the principles explained in the previous chapter.

The other type of gearing whose design involves the principles of Descriptive Geometry is the elliptical gear which works with a gear of two or more lobes. The axes of the gears are parallel, so that the pitch surfaces are cylinders (not cylinders of revolution). Since all right sections of cylinders are equal, this problem may be simplified by considering a section, in which case the pitch surfaces become pitch lines.

It can be shown that an ellipse formed by a plane section of a cone of revolution may be rolled on the development of the cone, the axis of rotation for the ellipse being taken through one focus, and for the cone development through the developed vertex. The ellipse and the cone development will therefore serve as pitch lines for the required gears. In order to have practicable gears, the pitch lines must form closed curves. That is, the development of the cone must occupy some aliquot part of 360°. In other words the gear which mates with the ellipse may consist of two lobes of 180° each, or three lobes of 120° each, etc. This means, according to the equation given in the chapter on developments, that the ratio of the slant height of the cone to the radius of the base must be a whole number and equal to the number of lobes on the gear.

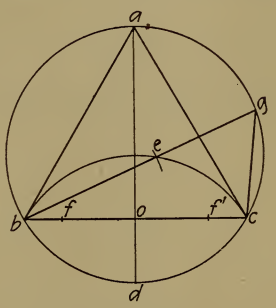


Fig. 187.

The following method, due to Prof. John B. Peddle, determines the position of the secant plane which will cut an ellipse of required major and minor axis from a cone of revolution.

Draw the triangle *abc*, Fig. 187, representing the cone

with vertex at *a* and base *bc*, the length *bc* being the major axis of the ellipse. The ratio of the slant height *ac* to the radius *oc* is made equal to the desired number of lobes. Circumscribe a circle about the triangle. Locate the foci of the ellipse, as *f* and *f'*. Produce *ao* to *d*, and draw a circular arc with *d* as center and *db* as radius. With center at *b* and radius *ff'* strike an arc to *e*. Join *be* and produce to *g*. The distances *gb* and *gc* will then be the longest and shortest elements of the cone.

The problem of properly shaping and locating the teeth on the pitch surfaces of a gear has no connection

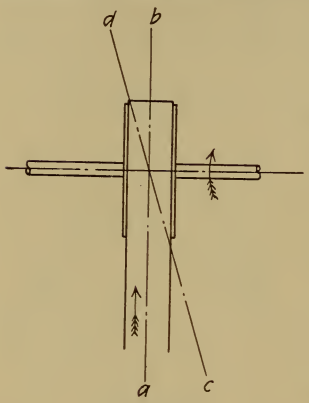


Fig. 188.

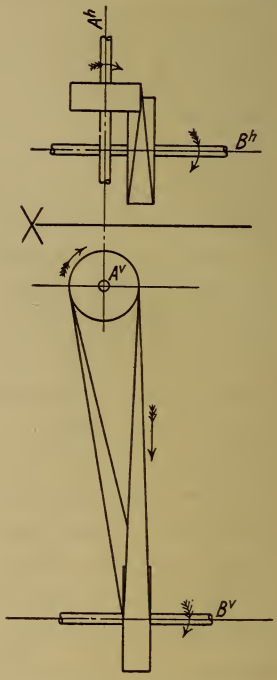


Fig. 189.

with Descriptive Geometry and will not be considered.

Belt Drive.—In the transmission of power by belts, it is necessary that the center line of the belt approaching a pulley must lie in the center plane of the pulley. In Fig. 188 the center plane of the pulley is represented by the line *ab*, and the center line of the belt approaching the pulley must lie along this line. If it were along some other line, as *cd*, the belt would run off the pulley. On the other side of

the pulley, where the belt leaves, it does not have to obey this law.

It is sometimes necessary to connect two shafts which are not parallel. This can be easily accomplished for one direction of rotation by properly locating the pulleys. One such case is illustrated in Fig. 189. It will be seen by tracing out the path of the belt that the center line of the belt approaching either pulley lies in the center plane of that pulley. If the direction of motion is reversed this is no longer true, and the belt will run off the pulleys. In order to make the drive reversible it is necessary to provide one or two guide pulleys, which will so guide the belt that it will run in either direction. The location of a single guide pulley to accomplish this is a problem in Descriptive Geometry.

As an illustration of this, assume that the shafts *A* and *B* in Fig. 190 are to be connected by a belt so as to run in the directions indicated, or with both directions reversed. The pulleys may be so placed that one side of the belt lies in the middle planes of both pulleys, and consequently will run in either direction. The other side of the belt will require a guide pulley, and this must be so placed that it will guide the portions of the belt on each side of it in the planes of the pulleys on *A* and *B*. The center lines of these two portions of the belt will be lines lying in the middle planes of the pulleys and intersecting in a point which is common to both the planes.

The first step in the solution of the problem, after the main pulleys have been located, is to select the position of this point lying in the planes of both pulleys. It may be taken at any convenient height, but not too close to either *A* or *B*. In Fig. 190, plane *3* is the mid plane of pulley *B*, plane *4* that of pulley *A*, and the point is chosen at *c*, on the line of intersection of *3* and *4*. From this point the two lines *C* and *D* are drawn, one tangent to each

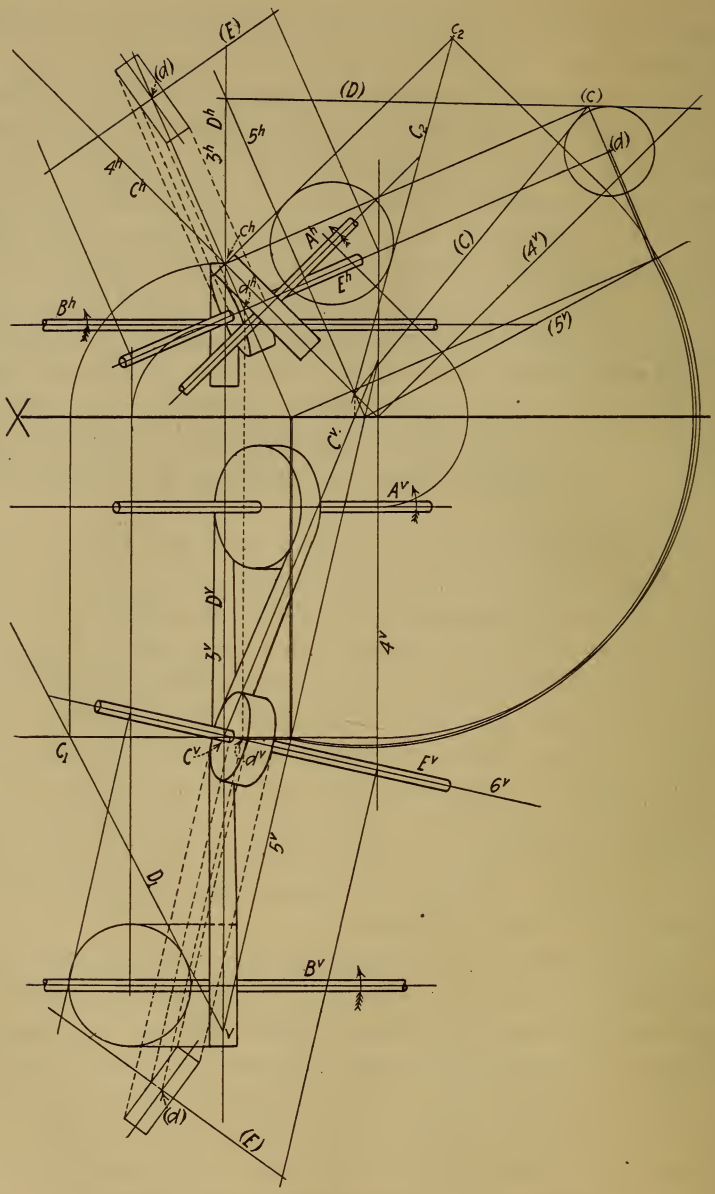

(E)
(d)
(D)
C2
(C)
D^h
5^h
3^h
4^h
C^h
C2
(d)
A^h
A1
E^h
(C)
(4^v)
B^h
(5^v)
C^v
A^v
D^v
3^v
4^v
C1
C^v
d^v
E^v
6^v
5^v
D1
B^v
V
(d)
(E)

Fig. 190.

pulley. This may or may not require the revolution of the mid planes of the pulleys into V or H, according as the pulleys are or are not oblique to V and H. In the figure, plane *3* has been revolved into V and plane *4* into H in order to draw lines *C* and *D*. The plane of the lines *C* and *D* is the plane of the guide pulley. Find the traces of this plane, as 5^h and 5^v. Revolve into H or V, and locate the revolved positions of point *c* and lines *C* and *D*. These are shown as (*c*), (*C*) and (*D*).

Assuming that the size of the guide pulley is known, it can be drawn tangent to (*C*) and (*D*) and its center point (*d*) located in revolved position. Counter revolve plane *5* and obtain d^h and d^v. The axis of the guide pulley shaft may then be drawn as line *E*, through *d* and perpendicular to plane *5*.

In order to find the projections of the guide pulley auxiliary planes may be used to advantage. To construct the V projection take plane *6* containing line *E* and perpendicular to V. Revolve it into V and construct the revolved position of the section cut from the pulley, which will be a rectangle. Counter revolve plane *6*, which gives the major and minor axes of the ellipses forming the required V projection. The H and P projections may be found by a similar construction.

Roof Truss Problems.—In the design of roof trusses a number of angles and dimensions may be found by the methods of Descriptive Geometry.[1] In steel trusses standard shapes such as I beams, angles, channels, etc., are used, and it is chiefly in the design of connecting angle plates that the methods of Descriptive Geometry offer convenient graphical solutions.

[1] Messrs. H. L. McKibben and L. E. Gray, in a book entitled "Hip and Valley Design" give graphical methods as well as analytic formula for the solution of a number of types of roof, hopper and pipe line designs.

Figure 191 shows the plan of a hip roof on a rectangular building. The lines marked A are the hip rafters and those marked B are the purlins. The purlins are so placed that their flanges lie in the plane of the roof and their webs are perpendicular to this plane. The hip rafters have their webs in a vertical plane. If the four planes of the roof are equally inclined to the horizontal, the hip rafter plans will be 45° lines.

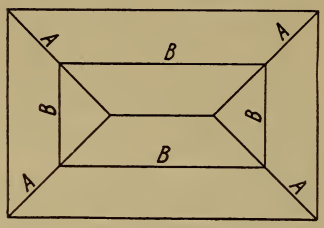


Fig. 191.

As an illustration of a problem relating to this type of roof, let it be required to design a bent plate for connecting a purlin to a hip rafter, the plate to be riveted to the web of each.

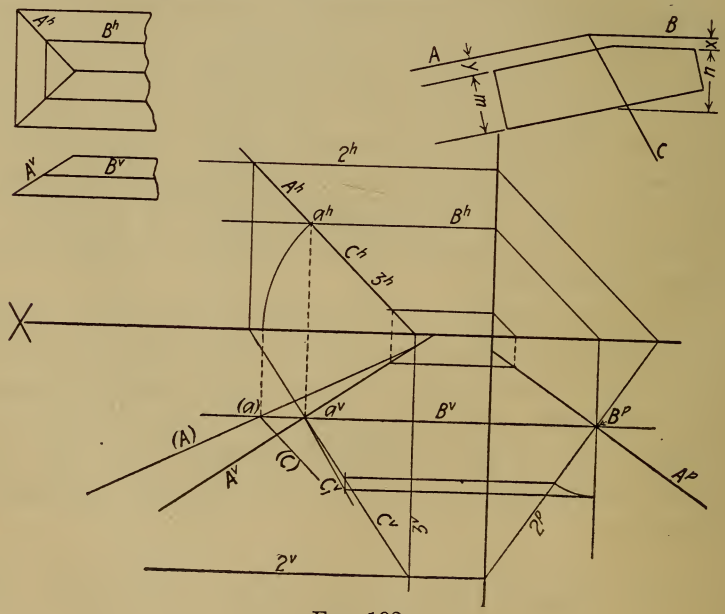


Fig. 192.

In Fig. 192, A is the hip rafter and B the purlin. The

plane of the hip rafter web is vertical, and its traces are 3^h and 3^v. The plane of the purlin web is perpendicular to the roof plane and its traces 2^h and 2^v may be found by means of a P projection, 2^p being perpendicular to A^p. The line C, which is the line of intersection of these two planes will be the line along which the connecting plate must be bent, and the angle between the planes is the angle to which the plate must be bent.

The shape of the connecting plate can be determined by revolving planes *2* and *3* into or parallel to V. The dimensions of the plate will be determined by the size and shape of the structural shapes used for hip rafter and purlin. Assuming these to be known we may proceed as follows: Revolve plane *3*, and with it lines A and C, into V. The lines in the revolved positions are shown as (A) and (C). Revolve plane *2* parallel to V about B as axis. Line B remains stationary and line C revolves to $C_1{}^v$. To determine the shape of the plate before bending draw any line representing B. Draw line C making with B the angle between B^v and $C_1{}^v$. Draw A making with C the angle between (A) and (C). A and B then represent the developed positions of these lines, and C the bend line.

It must be remembered that A and B lie in the plane of the roof, while the connecting plate must be at a distance from the roof plane so as to make allowance for the flanges of the rafter and purlin. These allowances, marked x and y in the figure, will be determined by the shape and dimensions of the structural shapes used. Draw parallels to A and B at these distances from them, and then lay off the required widths m and n of the two portions of the plate. These lines show the limiting lines for the required plate, and any outline lying inside of these, and bent along the line C will serve the purpose, provided sufficient plate area is provided for the required rivet holes. The angle to which the plate must be bent is the angle between planes *2*

and *3*, and may be determined by the methods of Problem 46. The projections of the bent plate can then be drawn.

Moulding Cutter Knives.—Wooden mouldings are cut by means of rotating knives, and the shape of the moulding cut depends upon the shape of the cutting edges of the knives. In general the shape of the moulding is not the same as that of the knife, because the axis of rotation does not lie in the plane of the knife. The knife, in revolving, sweeps out a solid of revolution, the meridian section of which is the shape of the moulding. If the plane of the knife contained the axis, then the shape of the knife and moulding would be the same.

If we know the shape of the moulding desired, the distance of the knife plane from the axis of rotation and also the distance of some particular point of the cutting edge from the axis, we can find the shape of the knife as follows: Draw a solid of revolution whose meridian section is the required moulding shape, making the particular point at the required distance from the axis. Cut this solid by a plane parallel to the axis and at a distance from it equal to the distance from the knife plane to the axis. This section is the required knife shape.

Mining Problems.—Deposits of ore frequently occur in beds which are essentially plane. The direction of a horizontal line in the ore bed is called the strike, and the angle of inclination of the bed with the horizontal is called the dip. The line along which the ore bed intersects the surface of the ground is called the outcrop. The locations and elevations of three points on the outcrop are sufficient data to determine the plane of the ore bed, and the various problems connected with this subject can be easily solved graphically.

To Find the Strike.—First locate both H and V projections of the three given points. Through one of them draw a

line parallel to H and lying in the plane of the three points. The H projection of this line gives the strike. This line can be most conveniently located without finding the traces of the plane. The V projection is drawn parallel to X. Join the other two points by a line, find both projections of the point where the horizontal line crosses it and join the H projection of this point with the H projection of the first point, which gives the H projection required.

To Find the Dip.—Take a P plane perpendicular to the horizontal line just found. Find the P projections of the three given points, and join them by a straight line. This represents the P projection or trace of the plane of the ore bed, and its inclination gives the required angle.

To Find the Depth of a Shaft.—Suppose the location and elevation of a fourth point on the surface of the ground is given, to find the depth of a shaft to strike the ore bed. Locate the P projection of this point. Its distance vertically over the ore bed is shown in the P projection.

For other problems and methods of this sort, the student is referred to the article "The Application of Descriptive Geometry to Mining Problems," by Joseph W. Roe, Transactions American Institute of Mining Engineers, Vol. XLI, page 512.

Exercises

381. Two horizontal shafts lie in planes 2 in. apart and their plans are inclined at 60°. It is desired to connect these shafts by skew bevel gearing whose speed ratio is 1 : 2. Find the radii of the gorge circles of the hyperboloids which will be the pitch surfaces of the gears, also the projections of the element of contact.

382. Show the projections of the hyperboloids of the preceding exercise.

383. A gear has a pitch line of elliptic form, major axis 3 in. and minor axis 2 in. Determine the pitch line of a bilobe gear which will work with the elliptic gear.

384. Determine the pitch line of a trilobe gear which will work with the elliptic gear of the preceding exercise.

385. Two horizontal shafts, whose center lines are shown in Fig. 193, are to be connected by a belt, and are to run in the directions indicated or both reversed. The pulley on shaft A is to be 18 in. in diameter, that on B 14 in., and the guide pulley 15 in. The guide pulley is to be located about halfway between the shafts. The width of belt is to be 3 in. Determine the location of the guide pulley and show projections of pulleys and belt. Scale, 1 in. = 1 ft. 0 in. This exercise requires a drawing space about 12 in. × 18 in. to include a P projection.

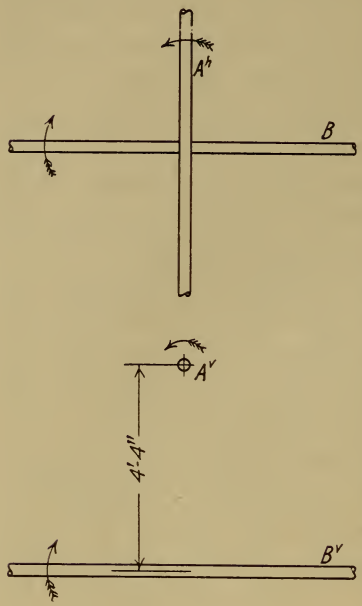


Fig. 193.

386. A rectangular building has a hip roof whose hip rafters are inclined at 30° to the horizontal. It is required to design a bent plate connection between purlin and hip rafter, given the following data: Connecting plate to be riveted to the webs of rafter and purlin. The rafter is a 6 in. I beam which will allow a plate 4 in. wide, and the purlin is a 4 in. channel, and since the plate is to be riveted to its back side, no allowance need be made for its flanges. Two ½-in. rivets are to be used in each portion of the plate, and center distances of rivets are to be at least 2 in. No rivet center is to come closer than 1 in. to the edge of the plate nor closer than 1½ in. to the bend line. The rivets on the channel must be at least 1½ in. from edge of channel, to allow for flanges. Draw the developed plate, locate bend line and determine bending angle. Scale, 3 in. = 1 ft. 0 in.

387. Find the profile for a cutter knife which shall cut a moulding of the section shown in Fig. 194. Assume that the plane of the knife is 1½ in. from the axis of rotation, and that the nearest point of the cutting edge is 2 in. from the axis.

388. Take the same data as in the preceding exercise, but the moulding section shown in Fig. 195.

389. Given three points a, b and c, on the outcrop of a bed of ore (Fig. 196). The elevations of the points are as follows:

a, 2920 ft.
b, 3060 ft.
c, 2840 ft.

Find the strike and dip. Scale, 1 in. = 80 ft.

390. Taking the same data as in the preceding exercise, find the

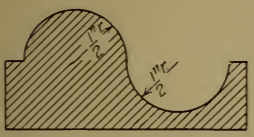


FIG. 194.

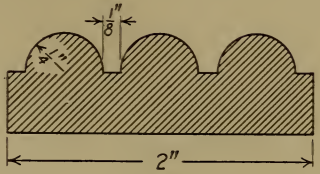


FIG. 195.

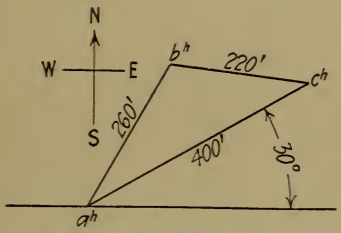


FIG. 196.

depth of a shaft at *d* to strike the ore bed. Point *d* is 250 ft. due east of *a*, and its elevation is 2890 ft.

391. Solve Exercise 390 by finding the traces of the plane $\overline{abc}$ and finding the point of intersection of the vertical shaft with this plane.

XXIII. Impossible and Indeterminate Exercises.

In the following an attempt should be made to solve the exercise. Then state clearly why the exercise is impossible or indeterminate.

392. Find the traces of the plane containing lines *C* and *D* (Fig. 75, page 46).

393. Find the traces of a plane containing line *A* and perpendicular to line *B* (Fig. 125, page 63).

394. Find a line of plane *2* making 60° with V (Fig. 118, page 60).

395. Find the traces of a plane which makes 30° with H and 45° with V.

396. Find the traces of the plane containing points *a*, *b* and *c* (Fig. 197).

397. Pass a plane through line *B* and parallel to line *A* (Fig. 119, page 60).

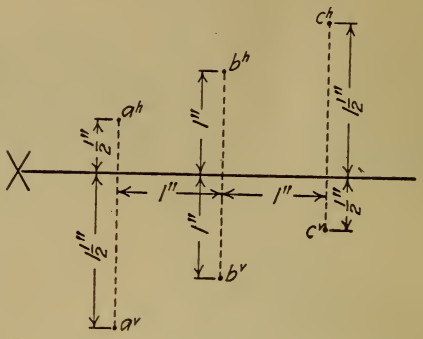


FIG. 197.

398. Find the projections of a line making 45° with H and 60° with V.

399. Through line *A* pass a plane making 30° with H (Fig. 119, page 60).

400. Show the traces of the plane containing point *a* and perpendicular to plane *2* (Fig. 118, page 60).

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